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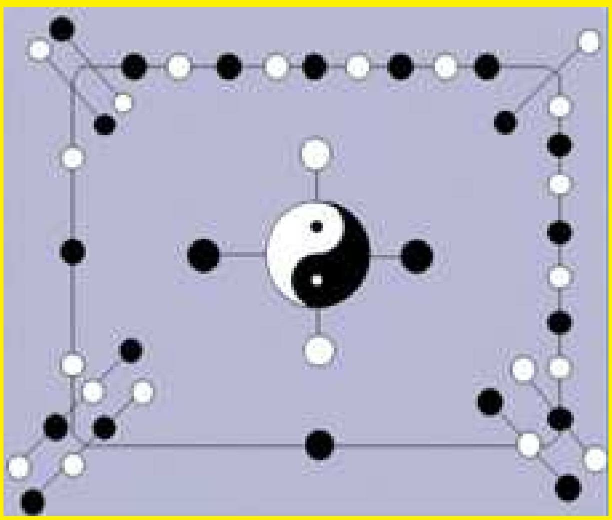
Edited By Linfan MAO



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(International Book Series)

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Linear Isometries on Pseudo-Euclidean Space $(\mathbb{R}^n, \mu)$

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Abstract: A *pseudo-Euclidean space* $(\mathbb{R}^n, \mu)$ is such a Euclidean space $\mathbb{R}^n$ associated with a mapping $\mu : \overrightarrow{V}_{\overline{x}} \to {}_{\overline{x}}\overrightarrow{V}$ for $\overline{x} \in \mathbb{R}^n$, and a linear isometry $T : (\mathbb{R}^n, \mu) \to (\mathbb{R}^n, \mu)$ is such a linear isometry $T : \mathbb{R}^n \to \mathbb{R}^n$ that $T\mu = \mu T$. In this paper, we characterize curvature of s-line, particularly, Smarandachely embedded graphs and determine linear isometries on $(\mathbb{R}^n, \mu)$.

Key Words: Smarandachely denied axiom, Smarandache geometry, s-line, pseudo-Euclidean space, isometry, Smarandachely map, Smarandachely embedded graph.

AMS(2010): 05C25, 05E15, 08A02, 15A03, 20E07, 51M15.

§1. Introduction

As we known, a Smarandache geometry is defined following.

Definition 1.1 *A rule $R \in \mathcal{R}$ in a mathematical system $(\Sigma; \mathcal{R})$ is said to be Smarandachely denied if it behaves in at least two different ways within the same set Σ, i.e., validated and invalided, or only invalided but in multiple distinct ways.*

Definition 1.2 *A Smarandache geometry is such a geometry in which there are at least one Smarandachely denied ruler and a Smarandache manifold $(M; \mathcal{A})$ is an n-dimensional manifold M that support a Smarandache geometry by Smarandachely denied axioms in $\mathcal{A}$. A line in a Smarandache geometry is called an s-line.*

Applying the structure of a Euclidean space $\mathbb{R}^n$, we are easily construct a special Smarandache geometry, called pseudo-Euclidean space([5]-[6]) following. Let $\mathbb{R}^n = \{(x_1, x_2, \cdots, x_n)\}$ be a Euclidean space of dimensional n with a normal basis $\overline{\epsilon}_1 = (1, 0, \cdots, 0)$, $\overline{\epsilon}_2 = (0, 1, \cdots, 0)$, $\cdots$, $\overline{\epsilon}_n = (0, 0, \cdots, 1)$, $\overline{x} \in \mathbb{R}^n$ and $\overrightarrow{V}_{\overline{x}}$, ${}_{\overline{x}}\overrightarrow{V}$ two vectors with end or initial point at $\overline{x}$, respectively. A *pseudo-Euclidean space* $(\mathbb{R}^n, \mu)$ is such a Euclidean space $\mathbb{R}^n$ associated with a mapping $\mu : \overrightarrow{V}_{\overline{x}} \to {}_{\overline{x}}\overrightarrow{V}$ for $\overline{x} \in \mathbb{R}^n$, such as those shown in Fig.1,

[1]Received October 8, 2011. Accepted February 4, 2012.

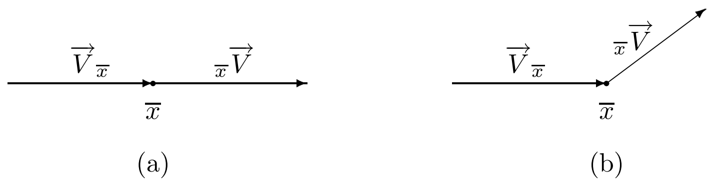

Fig.1

where $\overrightarrow{V}_{\overline{x}}$ and $_{\overline{x}}\overrightarrow{V}$ are in the same orientation in case (a), but not in case (b). Such points in case (a) are called *Euclidean* and in case (b) *non-Euclidean*. A pseudo-Euclidean $(\mathbb{R}^n, \mu)$ is *finite* if it only has finite non-Euclidean points, otherwise, *infinite*.

By definition, a Smarandachely denied axiom $A \in \mathcal{A}$ can be considered as an action of A on subsets $S \subset M$, denoted by S^A. If $(M_1; \mathcal{A}_1)$ and $(M_2; \mathcal{A}_2)$ are two Smarandache manifolds, where $\mathcal{A}_1$, $\mathcal{A}_2$ are the Smarandachely denied axioms on manifolds M_1 and M_2, respectively. They are said to be *isomorphic* if there is $1-1$ mappings $\tau : M_1 \to M_2$ and $\sigma : \mathcal{A}_1 \to \mathcal{A}_2$ such that $\tau(S^A) = \tau(S)^{\sigma(\mathcal{A})}$ for $\forall S \subset M_1$ and $A \in \mathcal{A}_1$. Such a pair (τ, σ) is called an isomorphism between $(M_1; \mathcal{A}_1)$ and $(M_2; \mathcal{A}_2)$. Particularly, if $M_1 = M_2 = M$ and $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}$, such an isomorphism (τ, σ) is called a *Smarandachely automorphism* of $(M, \mathcal{A})$. Clearly, all such automorphisms of $(M, \mathcal{A})$ form an group under the composition operation on τ for a given σ. Denoted by $\mathrm{Aut}(M, \mathcal{A})$. A special Smarandachely automorphism, i.e., linear isomorphism on a pseudo-Euclidean space $(\mathbb{R}^n, \mu)$ is defined following.

Definition 1.3 *Let $(\mathbb{R}^n, \mu)$ be a pseudo-Euclidean space with normal basis $\{\overline{\epsilon}_1, \overline{\epsilon}_2, \cdots, \overline{\epsilon}_n\}$. A linear isometry $T : (\mathbb{R}^n, \mu) \to (\mathbb{R}^n, \mu)$ is such a transformation that*

$$T(c_1\overline{e}_1 + c_2\overline{e}_2) = c_1T(\overline{e}_1) + c_2T(\overline{e}_2), \quad \langle T(\overline{e}_1), T(\overline{e}_2)\rangle = \langle \overline{e}_1, \overline{e}_2\rangle \quad and \quad T\mu = \mu T$$

for $\overline{e}_1,\ \overline{e}_2 \in \mathbf{E}$ and $c_1,\ c_2 \in \mathscr{F}$.

Denoted by $\mathrm{Isom}(\mathbb{R}^n, \mu)$ the set of all linear isometries of $(\mathbb{R}^n, \mu)$. Clearly, $\mathrm{Isom}(\mathbb{R}^n, \mu)$ is a subgroup of $\mathrm{Aut}(M, \mathcal{A})$.

By definition, determining automorphisms of a Smarandache geometry is dependent on the structure of manifold M and axioms $\mathcal{A}$. So it is hard in general even for a manifold. The main purpose of this paper is to determine linear isometries and characterize the behavior of s-lines, particularly, Smarandachely embedded graphs in pseudo-Euclidean spaces $(\mathbb{R}^n, \mu)$. For terminologies and notations not defined in this paper, we follow references [1] for permutation group, [2]-[4] and [7]-[8] for graph, map and Smarandache geometry.

§2. Smarandachely Embedded Graphs in $(\mathbb{R}^n, \mu)$

2.1 Smarandachely Planar Maps

Let L be an s-line in a Smarandache plane $(\mathbf{R}^2, \mu)$ with non-Euclisedn points $A_1, A_2, \cdots, A_m$ for an integer $m \geq 0$. Its *curvature* $R(L)$ is defined by

$$R(L) = \sum_{i=1}^{m} (\pi - \mu(A_i)).$$

An s-line L is called *Euclidean* or *non-Euclidean* if $R(L) = \pm 2\pi$ or $\neq \pm 2\pi$. The following result characterizes s-lines on $(\mathbf{R}^2, \mu)$.

Theorem 2.1 *An s-line without self-intersections is closed if and only if it is Euclidean.*

Proof Let $(\mathbf{R}^2, \mu)$ be a Smarandache plane and let L be a closed s-line without self-intersections on $(\mathbf{R}^2, \mu)$ with vertices $A_1, A_2, \cdots, A_m$. From the Euclid geometry on plane, we know that the angle sum of an m-polygon is $(m-2)\pi$. Whence, the curvature $R(L)$ of s-line L is $\pm 2\pi$ by definition, i.e., L is Euclidean.

Now if an s-line L is Euclidean, then $R(L) = \pm 2\pi$ by definition. Thus there exist non-Euclidean points $B_1, B_2, \cdots, B_m$ such that

$$\sum_{i=1}^{m} (\pi - \mu(B_i)) = \pm 2\pi.$$

Whence, L is nothing but an n-polygon with vertices $B_1, B_2, \cdots, B_m$ on $\mathbf{R}^2$. Therefore, L is closed without self-intersection. □

A planar map is a 2-cell embedding of a graph G on Euclidean plane $\mathbb{R}^2$. It is called *Smarandachely* on $(\mathbb{R}^2, \mu)$ if all of its vertices are elliptic (hyperbolic). Notice that these pendent vertices is not important because it can be always Euclidean or non-Euclidean. We concentrate our attention to non-separated maps. Such maps always exist circuit-decompositions. The following result characterizes Smarandachely planar maps.

Theorem 2.2 *A non-separated planar map M is Smarandachely if and only if there exist a directed circuit-decomposition*

$$E_{\frac{1}{2}}(M) = \bigoplus_{i=1}^{s} E(\overrightarrow{C}_i)$$

of M such that one of the linear systems of equations

$$\sum_{v \in V(\overrightarrow{C}_i)} (\pi - x_v) = 2\pi, \quad or \quad \sum_{v \in V(\overrightarrow{C}_i)} (\pi - x_v) = -2\pi, \quad 1 \leq i \leq s$$

is solvable, where $E_{\frac{1}{2}}(M)$ denotes the set of semi-arcs of M.

Proof If M is Smarandachely, then each vertex $v \in V(M)$ is non-Euclidean, i.e., $\mu(v) \neq \pi$. Whence, there exists a directed circuit-decomposition

$$E_{\frac{1}{2}}(M) = \bigoplus_{i=1}^{s} E(\overrightarrow{C}_i)$$

of semi-arcs in M such that each of them is an s-line in $(\mathbf{R}^2, \mu)$. Applying Theorem 9.3.5, we know that

$$\sum_{v \in V(\overrightarrow{C}_i)} (\pi - \mu(v)) = 2\pi \quad \text{or} \quad \sum_{v \in V(\overrightarrow{C}_i)} (\pi - \mu(v)) = -2\pi$$

for each circuit $C_i,\ 1 \leq i \leq s$. Thus one of the linear systems of equations

$$\sum_{v \in V(\overrightarrow{C}_i)} (\pi - x_v) = 2\pi, \quad 1 \leq i \leq s \quad \text{or} \quad \sum_{v \in V(\overrightarrow{C}_i)} (\pi - x_v) = -2\pi, \quad 1 \leq i \leq s$$

is solvable.

Conversely, if one of the linear systems of equations

$$\sum_{v \in V(\overrightarrow{C}_i)} (\pi - x_v) = 2\pi, \quad 1 \leq i \leq s \quad \text{or} \quad \sum_{v \in V(\overrightarrow{C}_i)} (\pi - x_v) = -2\pi, \quad 1 \leq i \leq s$$

is solvable, define a mapping $\mu : \mathbf{R}^2 \to [0, 4\pi)$ by

$$\mu(x) = \begin{cases} x_v & \text{if } x = v \in V(M), \\ \pi & \text{if } x \notin v(M). \end{cases}$$

Then M is a Smarandachely map on $(\mathbf{R}^2, \mu)$. This completes the proof. $\square$

In Fig.2, we present an example of a Smarandachely planar maps with μ defined by numbers on vertices.

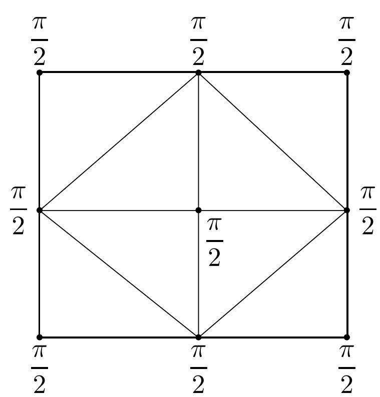

Fig.2

Let $\omega_0 \in (0, \pi)$. An s-line L is called *non-Euclidean of type* ω_0 if $R(L) = \pm 2\pi \pm \omega_0$. Similar to Theorem 2.2, we can get the following result.

Theorem 2.3 *A non-separated map M is Smarandachely if and only if there exist a directed circuit-decomposition*

$$E_{\frac{1}{2}}(M) = \bigoplus_{i=1}^{s} E(\overrightarrow{C}_i)$$

of M into s-lines of type ω_0, $\omega_0 \in (0, \pi)$ for integers $1 \leq i \leq s$ such that one of the linear

systems of equations

$$\sum_{v\in V(\overrightarrow{C}_i)} (\pi - x_v) = 2\pi - \omega_0, \qquad 1 \le i \le s,$$
$$\sum_{v\in V(\overrightarrow{C}_i)} (\pi - x_v) = -2\pi - \omega_0, \qquad 1 \le i \le s,$$
$$\sum_{v\in V(\overrightarrow{C}_i)} (\pi - x_v) = 2\pi + \omega_0, \qquad 1 \le i \le s,$$
$$\sum_{v\in V(\overrightarrow{C}_i)} (\pi - x_v) = -2\pi + \omega_0, \qquad 1 \le i \le s$$

is solvable.

2.2 Smarandachely Embedded Graphs in $(\mathbb{R}^n, \mu)$

Generally, we define the *curvature* $R(L)$ of an s-line L passing through non-Euclidean points $\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_m \in \mathbf{R}^n$ for $m \ge 0$ in $(\mathbf{R}^n, \mu)$ to be a matrix determined by

$$R(L) = \prod_{i=1}^{m} \mu(\overline{x}_i)$$

and *Euclidean* if $R(L) = I_{n\times n}$, otherwise, *non-Euclidean*. It is obvious that a point in a Euclidean space $\mathbf{R}^n$ is indeed Euclidean by this definition. Furthermore, we immediately get the following result for Euclidean s-lines in $(\mathbf{R}^n, \mu)$.

Theorem 2.4 *Let* $(\mathbf{R}^n, \mu)$ *be a pseudo-Euclidean space and* L *an s-line in* $(\mathbf{R}^n, \mu)$ *passing through non-Euclidean points* $\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_m \in \mathbf{R}^n$*. Then* L *is closed if and only if* L *is Euclidean.*

Proof If L is a closed s-line, then L is consisted of vectors $\overrightarrow{\overline{x}_1\overline{x}_2}$, $\overrightarrow{\overline{x}_2\overline{x}_3}$, $\cdots$, $\overrightarrow{\overline{x}_n\overline{x}_1}$. By definition,

$$\frac{\overrightarrow{\overline{x}_{i+1}\overline{x}_i}}{\left|\overrightarrow{\overline{x}_{i+1}\overline{x}_i}\right|} = \frac{\overrightarrow{\overline{x}_{i-1}\overline{x}_i}}{\left|\overrightarrow{\overline{x}_{i-1}\overline{x}_i}\right|}\ \mu(\overline{x}_i)$$

for integers $1 \le i \le m$, where $i + 1 \equiv (\bmod m)$. Consequently,

$$\overrightarrow{\overline{x}_1\overline{x}_2} = \overrightarrow{\overline{x}_1\overline{x}_2} \prod_{i=1}^{m}\ \mu(\overline{x}_i).$$

Thus $\prod\limits_{i=1}^{m} \mu(\overline{x}_i) = I_{n\times n}$, i.e., L is Euclidean.

Conversely, let L be Euclidean, i.e., $\prod\limits_{i=1}^{m}\ \mu(\overline{x}_i) = I_{n\times n}$. By definition, we know that

$$\frac{\overrightarrow{\overline{x}_{i+1}\overline{x}_i}}{\left|\overrightarrow{\overline{x}_{i+1}\overline{x}_i}\right|} = \frac{\overrightarrow{\overline{x}_{i-1}\overline{x}_i}}{\left|\overrightarrow{\overline{x}_{i-1}\overline{x}_i}\right|}\ \mu(\overline{x}_i), \quad \text{i.e.,} \quad \overrightarrow{\overline{x}_{i+1}\overline{x}_i} = \frac{\left|\overrightarrow{\overline{x}_{i+1}\overline{x}_i}\right|}{\left|\overrightarrow{\overline{x}_{i-1}\overline{x}_i}\right|}\overrightarrow{\overline{x}_{i-1}\overline{x}_i}\ \mu(\overline{x}_i)$$

for integers $1 \leq i \leq m$, where $i+1 \equiv (\bmod m)$. Whence, if $\prod\limits_{i=1}^{m} \mu(\overline{x}_i) = I_{n\times n}$, then there must be

$$\overrightarrow{\overline{x}_1\overline{x}_2} = \overrightarrow{\overline{x}_1\overline{x}_2}\prod_{i=1}^{m} \mu(\overline{x}_i).$$

Thus L consisted of vectors $\overrightarrow{\overline{x}_1\overline{x}_2}$, $\overrightarrow{\overline{x}_2\overline{x}_3}$, $\cdots$, $\overrightarrow{\overline{x}_n\overline{x}_1}$ is a closed s-line in $(\mathbf{R}^n, \mu)$. $\square$

Now we consider the pseudo-Euclidean space $(\mathbf{R}^2, \mu)$ and find the rotation matrix $\mu(\overline{x})$ for points $\overline{x} \in \mathbf{R}^2$. Let $\theta_{\overline{x}}$ be the angle form $\overline{\epsilon}_1$ to $\mu\overline{\epsilon}_1$. Then it is easily to know that

$$\mu(\overline{x}) = \begin{pmatrix} \cos\theta_{\overline{x}} & \sin\theta_{\overline{x}} \\ \sin\theta_{\overline{x}} & -\cos\theta_{\overline{x}} \end{pmatrix}.$$

Now if an s-line L passing through non-Euclidean points $\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_m \in \mathbf{R}^2$, then Theorem 2.4 implies that

$$\begin{pmatrix} \cos\theta_{\overline{x}_1} & \sin\theta_{\overline{x}_1} \\ \sin\theta_{\overline{x}_1} & -\cos\theta_{\overline{x}_1} \end{pmatrix}\begin{pmatrix} \cos\theta_{\overline{x}_2} & \sin\theta_{\overline{x}_2} \\ \sin\theta_{\overline{x}_2} & -\cos\theta_{\overline{x}_2} \end{pmatrix}\cdots\begin{pmatrix} \cos\theta_{\overline{x}_m} & \sin\theta_{\overline{x}_m} \\ \sin\theta_{\overline{x}_m} & -\cos\theta_{\overline{x}_m} \end{pmatrix} = I_{2\times 2}.$$

Thus

$$\mu(\overline{x}) = \begin{pmatrix} \cos(\theta_{\overline{x}_1}+\theta_{\overline{x}_2}+\cdots+\theta_{\overline{x}_m}) & \sin(\theta_{\overline{x}_1}+\theta_{\overline{x}_2}+\cdots+\theta_{\overline{x}_m}) \\ \sin(\theta_{\overline{x}_1}+\theta_{\overline{x}_2}+\cdots+\theta_{\overline{x}_m}) & \cos(\theta_{\overline{x}_1}+\theta_{\overline{x}_2}+\cdots+\theta_{\overline{x}_m}) \end{pmatrix} = I_{2\times 2}.$$

Whence, $\theta_{\overline{x}_1}+\theta_{\overline{x}_2}+\cdots+\theta_{\overline{x}_m} = 2k\pi$ for an integer k. This fact is in agreement with that of Theorem 2.1, only with different disguises.

An *embedded graph* G on $\mathbf{R}^n$ is a $1-1$ mapping $\tau : G \to \mathbf{R}^n$ such that for $\forall e, e' \in E(G)$, $\tau(e)$ has no self-intersection and $\tau(e)$, $\tau(e')$ maybe only intersect at their end points. Such an embedded graph G in $\mathbf{R}^n$ is denoted by $G_{\mathbf{R}^n}$. For example, the n-cube $\mathcal{C}_n$ is such an embedded graph with vertex set $V(\mathcal{C}_n) = \{\ (x_1, x_2, \cdots, x_n) \mid x_i = 0 \ \text{ or } \ 1 \ \ for\ 1 \leq i \leq n\ \}$ and two vertices $(x_1, x_2, \cdots, x_n))$ and $(x_1', x_2', \cdots, x_n')$ are adjacent if and only if they are differ exactly in one entry. We present two n-cubes in Fig.3 for $n = 2$ and $n = 3$.

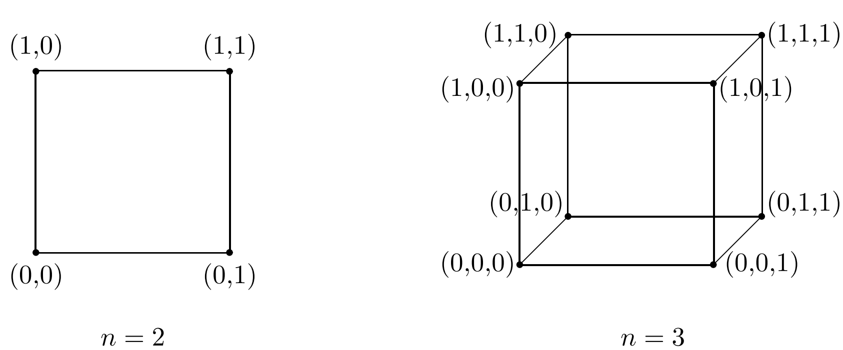

Fig.3

Similarly, an embedded graph $G_{\mathbf{R}^n}$ is called *Smarandachely* if there exists a pseudo-Euclidean space $(\mathbf{R}^n, \mu)$ with a mapping $\mu : \overline{x} \in \mathbf{R}^n \to [\overline{x}]$ such that all of its vertices are non-Euclidean points in $(\mathbf{R}^n, \mu)$. Certainly, these vertices of valency 1 is not important for Smarandachely embedded graphs. We concentrate our attention on embedded 2-connected graphs.

Theorem 2.5 *An embedded* 2*-connected graph* $G_{\mathbf{R}^n}$ *is Smarandachely if and only if there is a mapping* $\mu : \overline{x} \in \mathbf{R}^n \to [\overline{x}]$ *and a directed circuit-decomposition*

$$E_{\frac{1}{2}} = \bigoplus_{i=1}^{s} E(\overrightarrow{C}_i)$$

such that these matrix equations

$$\prod_{\overline{x} \in V(\overrightarrow{C}_i)} X_{\overline{x}} = I_{n \times n} \quad 1 \le i \le s$$

are solvable.

Proof By definition, if $G_{\mathbf{R}^n}$ is Smarandachely, then there exists a mapping $\mu : \overline{x} \in \mathbf{R}^n \to [\overline{x}]$ on $\mathbf{R}^n$ such that all vertices of $G_{\mathbf{R}^n}$ are non-Euclidean in $(\mathbf{R}^n, \mu)$. Notice there are only two orientations on an edge in $E(G_{\mathbf{R}^n})$. Traveling on $G_{\mathbf{R}^n}$ beginning from any edge with one orientation, we get a closed s-line $\overrightarrow{C}$, i.e., a directed circuit. After we traveled all edges in $G_{\mathbf{R}^n}$ with the possible orientations, we get a directed circuit-decomposition

$$E_{\frac{1}{2}} = \bigoplus_{i=1}^{s} E(\overrightarrow{C}_i)$$

with an s-line $\overrightarrow{C}_i$ for integers $1 \le i \le s$. Applying Theorem 2.4, we get

$$\prod_{\overline{x} \in V(\overrightarrow{C}_i)} \mu(\overline{x}) = I_{n \times n} \quad 1 \le i \le s.$$

Thus these equations

$$\prod_{\overline{x} \in V(\overrightarrow{C}_i)} X_{\overline{x}} = I_{n \times n} \quad 1 \le i \le s$$

have solutions $X_{\overline{x}} = \mu(\overline{x})$ for $\overline{x} \in V(\overrightarrow{C}_i)$.

Conversely, if these is a directed circuit-decomposition

$$E_{\frac{1}{2}} = \bigoplus_{i=1}^{s} E(\overrightarrow{C}_i)$$

such that these matrix equations

$$\prod_{\overline{x} \in V(\overrightarrow{C}_i)} X_{\overline{x}} = I_{n \times n} \quad 1 \le i \le s$$

are solvable, let $X_{\overline{x}} = A_{\overline{x}}$ be such a solution for $\overline{x} \in V(\overrightarrow{C}_i)$, $1 \le i \le s$. Define a mapping $\mu : \overline{x} \in \mathbf{R}^n \to [\overline{x}]$ on $\mathbf{R}^n$ by

$$\mu(\overline{x}) = \begin{cases} A_{\overline{x}} & \text{if } \overline{x} \in V(G_{\mathbf{R}^n}), \\ I_{n\times n} & \text{if } \overline{x} \notin V(G_{\mathbf{R}^n}). \end{cases}$$

Then we get a Smarandachely embedded graph $G_{\mathbf{R}^n}$ in the pseudo-Euclidean space $(\mathbf{R}^n, \mu)$ by Theorem 2.4. $\square$

§3. Linear Isometries on Pseudo-Euclidean Space

If all points in a pseudo-Euclidean space $(\mathbb{R}^n, \mu)$ are Euclidean, i.e., the case (a) in Fig.1, then $(\mathbb{R}^n, \mu)$ is nothing but just the Euclidean space $\mathbf{R}^n$. The following results on linear isometries of Euclidean spaces are well-known.

Theorem 3.1 *Let $\mathbf{E}$ be an n-dimensional Euclidean space with normal basis $\{\overline{\epsilon}_1, \overline{\epsilon}_2, \cdots, \overline{\epsilon}_n\}$ and T a linear transformation on $\mathbf{E}$ determined by $\overline{Y}^t = [a_{ij}]_{n\times n} \overline{X}^t$, where $\overline{X} = (\overline{\epsilon}_1, \overline{\epsilon}_2, \cdots, \overline{\epsilon}_n)$ and $\overline{Y} = (T(\overline{\epsilon}_1), T(\overline{\epsilon}_2), \cdots, T(\overline{\epsilon}_n))$. Then T is a linear isometry on $\mathbf{E}$ if and only if $[a_{ij}]_{n\times n}$ is an orthogonal matrix, i.e., $[a_{ij}]_{n\times n} [a_{ij}]^t_{n\times n} = I_{n\times n}$.*

Theorem 3.2 *An isometry on a Euclidean space $\mathbf{E}$ is a composition of three elementary isometries on $\mathbf{E}$ following:*

Translation $\mathbb{T}_{\overline{e}}$. *A mapping that moves every point $(x_1, x_2, \cdots, x_n)$ of $\mathbf{E}$ by*

$$T_{\overline{e}} : (x_1, x_2, \cdots, x_n) \to (x_1 + e_1, x_2 + e_2, \cdots, x_n + e_n),$$

where $\overline{e} = (e_1, e_2, \cdots, e_n)$.

Rotation $\mathbb{R}_{\overline{\theta}}$. *A mapping that moves every point of $\mathbf{E}$ through a fixed angle about a fixed point. Similarly, taking the center O to be the origin of polar coordinates $(r, \phi_1, \phi_2, \cdots, \phi_{n-1})$, a rotation $R_{\theta_1,\theta_2,\cdots,\theta_{n-1}} : \mathbf{E} \to \mathbf{E}$ is*

$$R_{\theta_1,\theta_2,\cdots,\theta_{n-1}} : (r, \phi_1, \phi_2, \cdots, \phi_{n_1}) \to (r, \phi_1 + \theta_1, \phi_2 + \theta_2, \cdots, \phi_{n_1} + \theta_{n-1}),$$

where θ_i is a constant angle, $\theta_i \in \mathbf{R}$ $(\mathrm{mod} 2\pi)$ for integers $1 \leq i \leq n-1$.

Reflection $\mathbb{F}$. *A reflection F is a mapping that moves every point of $\mathbf{E}$ to its mirror-image in a fixed Euclidean subspace E' of dimensional $n-1$, denoted by $F = F(E')$. Thus for a point P in $\mathbf{E}$, $F(P) = P$ if $P \in E'$, and if $P \notin E'$, then $F(P)$ is the unique point in $\mathbf{E}$ such that E' is the perpendicular bisector of P and $F(P)$.*

Theorem 3.3 *An isometry $\mathcal{I}$ on a Euclidean space $\mathbf{E}$ is affine, i.e., determined by*

$$\overline{Y}^t = \lambda [a_{ij}]_{n\times n} \overline{X}^t + \overline{e},$$

where λ is a constant number, $[a_{ij}]_{n\times n}$ a orthogonal matrix and $\overline{e}$ a constant vector in $\mathbf{E}$.

Notice that a vector $\overrightarrow{V}$ can be uniquely determined by the basis of $\mathbf{R}^n$. For $\overline{x} \in \mathbf{R}^n$, there are infinite orthogonal frames at point $\overline{x}$. Denoted by $\mathcal{O}_{\overline{x}}$ the set of all normal bases at

point $\overline{x}$. Then a *pseudo-Euclidean space* $(\mathbf{R}, \mu)$ is nothing but a Euclidean space $\mathbf{R}^n$ associated with a linear mapping $\mu : \{\overline{\epsilon}_1, \overline{\epsilon}_2, \cdots, \overline{\epsilon}_n\} \to \{\overline{\epsilon}'_1, \overline{\epsilon}'_2, \cdots, \overline{\epsilon}'_n\} \in \mathcal{O}_{\overline{x}}$ such that $\mu(\overline{\epsilon}_1) = \overline{\epsilon}'_1$, $\mu(\overline{\epsilon}_2) = \overline{\epsilon}'_2$, $\cdots$, $\mu(\overline{\epsilon}_n) = \overline{\epsilon}'_n$ at point $\overline{x} \in \mathbf{R}^n$. Thus if $\overrightarrow{V}_{\overline{x}} = c_1\overline{\epsilon}_1 + c_2\overline{\epsilon}_2 + \cdots + c_n\overline{\epsilon}_n$, then $\mu({}_{\overline{x}}\overrightarrow{V}) = c_1\mu(\overline{\epsilon}_1) + c_2\mu(\overline{\epsilon}_2) + \cdots + c_n\mu(\overline{\epsilon}_n) = c_1\overline{\epsilon}'_1 + c_2\overline{\epsilon}'_2 + \cdots + c_n\overline{\epsilon}'_n$.

Without loss of generality, assume that

$$\begin{aligned}
&\mu(\overline{\epsilon}_1) = x_{11}\overline{\epsilon}_1 + x_{12}\overline{\epsilon}_2 + \cdots + x_{1n}\overline{\epsilon}_n,\\
&\mu(\overline{\epsilon}_2) = x_{21}\overline{\epsilon}_1 + x_{22}\overline{\epsilon}_2 + \cdots + x_{2n}\overline{\epsilon}_n,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
&\mu(\overline{\epsilon}_n) = x_{n1}\overline{\epsilon}_1 + x_{n2}\overline{\epsilon}_2 + \cdots + x_{nn}\overline{\epsilon}_n.
\end{aligned}$$

Then we find that

$$\begin{aligned}
\mu({}_{\overline{x}}\overrightarrow{V}) &= (c_1, c_2, \cdots, c_n)(\mu(\overline{\epsilon}_1), \mu(\overline{\epsilon}_2), \cdots, \mu(\overline{\epsilon}_n))^t\\
&= (c_1, c_2, \cdots, c_n)\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n}\\ x_{21} & x_{22} & \cdots & x_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ x_{n1} & x_{n2} & \cdots & x_{nn}\end{pmatrix}(\overline{\epsilon}_1, \overline{\epsilon}_2, \cdots, \overline{\epsilon}_n)^t.
\end{aligned}$$

Denoted by

$$[\overline{x}] = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n}\\ x_{21} & x_{22} & \cdots & x_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ x_{n1} & x_{n2} & \cdots & x_{nn}\end{pmatrix} = \begin{pmatrix} \langle\mu(\overline{\epsilon}_1), \overline{\epsilon}_1\rangle & \langle\mu(\overline{\epsilon}_1), \overline{\epsilon}_2\rangle & \cdots & \langle\mu(\overline{\epsilon}_1), \overline{\epsilon}_n\rangle\\ \langle\mu(\overline{\epsilon}_2), \overline{\epsilon}_1\rangle & \langle\mu(\overline{\epsilon}_2), \overline{\epsilon}_2\rangle & \cdots & \langle\mu(\overline{\epsilon}_2), \overline{\epsilon}_n\rangle\\ \cdots & \cdots & \cdots & \cdots\\ \langle\mu(\overline{\epsilon}_n), \overline{\epsilon}_1\rangle & \langle\mu(\overline{\epsilon}_n), \overline{\epsilon}_2\rangle & \cdots & \langle\mu(\overline{\epsilon}_n), \overline{\epsilon}_n\rangle\end{pmatrix},$$

called the *rotation matrix* of $\overline{x}$ in $(\mathbf{R}^n, \mu)$. Then $\mu : \overrightarrow{V}_{\overline{x}} \to {}_{\overline{x}}\overrightarrow{V}$ is determined by $\mu(\overline{x}) = [\overline{x}]$ for $\overline{x} \in \mathbf{R}^n$. Furthermore, such an rotation matrix $[\overline{x}]$ is orthogonal for points $\overline{x} \in \mathbf{R}^n$ by definition, i.e., $[\overline{x}]\,[\overline{x}]^t = I_{n\times n}$. Particularly, if $\overline{x}$ is Euclidean, then such an orientation matrix is nothing but $\mu(\overline{x}) = I_{n\times n}$. Summing up all these discussions, we know the following result.

Theorem 3.4 *If $(\mathbf{R}^n, \mu)$ is a pseudo-Euclidean space, then $\mu(\overline{x}) = [\overline{x}]$ is an $n \times n$ orthogonal matrix for $\forall\ \overline{x} \in \mathbf{R}^n$.*

By definition, we know that $\mathrm{Isom}(\mathbf{R}^n) = \langle \mathbb{T}_{\overline{e}}, \mathbb{R}_{\overline{\theta}}, \mathbb{F}\rangle$. An isometry τ of a pseudo-Euclidean space $(\mathbf{R}^n, \mu)$ is an isometry on $\mathbf{R}^n$ such that $\mu(\tau(\overline{x})) = \mu(\overline{x})$ for $\forall\overline{x} \in \mathbf{R}^n$. Clearly, all such isometries form a group $\mathrm{Isom}(\mathbf{R}^n, \mu)$ under composition operation with $\mathrm{Isom}(\mathbf{R}^n, \mu) \leq \mathrm{Isom}(\mathbf{R}^n)$. We determine isometries of pseudo-Euclidean spaces in this subsection.

Certainly, if $\mu(\overline{x})$ is a constant matrix $[c]$ for $\forall\overline{x} \in \mathbf{R}^n$, then all isometries on $\mathbf{R}^n$ is also isometries on $(\mathbf{R}^n, \mu)$. Whence, we only discuss those cases with at least two values for $\mu : \overline{x} \in \mathbf{R}^n \to [\overline{x}]$ similar to that of $(\mathbf{R}^2, \mu)$.

Translation. Let $(\mathbf{R}^n, \mu)$ be a pseudo-Euclidean space with an isometry of translation $T_{\overline{e}}$, where $\overline{e} = (e_1, e_2, \cdots, e_n)$ and $P,\ Q \in (\mathbf{R}^n, \mu)$ a non-Euclidean point, a Euclidean point,

respectively. Then $\mu(T_{\overline{e}}^k(P)) = \mu(P)$, $\mu(T_{\overline{e}}^k(Q)) = \mu(Q)$ for any integer $k \geq 0$ by definition. Consequently,

$$P,\ T_{\overline{e}}(P),\ T_{\overline{e}}^2(P),\ \cdots,\ T_{\overline{e}}^k(P), \cdots,$$
$$Q,\ T_{\overline{e}}(Q),\ T_{\overline{e}}^2(Q),\ \cdots,\ T_{\overline{e}}^k(Q), \cdots$$

are respectively infinite non-Euclidean and Euclidean points. Thus there are no isometries of translations if $(\mathbf{R}^n, \mu)$ is finite.

In this case, if there are rotations $R_{\theta_1,\theta_2,\cdots,\theta_{n-1}}$, then there must be $\theta_1, \theta_2, \cdots, \theta_{n-1} \in \{0, \pi/2\}$ and if $\theta_i = \pi/2$ for $1 \leq i \leq l$, $\theta_i = 0$ if $i \geq l+1$, then $e_1 = e_2 = \cdots = e_{l+1}$.

Rotation. Let (R^n, μ) be a pseudo-Euclidean space with an isometry of rotation $R_{\theta_1,\cdots,\theta_{n-1}}$ and $P,\ Q \in (\mathbf{R}^n, \mu)$ a non-Euclidean point, a Euclidean point, respectively. Then

$$\mu(R_{\theta_1,\theta_2,\cdots,\theta_{n-1}}(P)) = \mu(P), \quad \mu(R_{\theta_1,\theta_2,\cdots,\theta_{n-1}}(Q)) = \mu(Q)$$

for any integer $k \geq 0$ by definition. Whence,

$$P,\ R_{\theta_1,\theta_2,\cdots,\theta_{n-1}}(P),\ R^2_{\theta_1,\theta_2,\cdots,\theta_{n-1}}(P),\ \cdots,\ R^k_{\theta_1,\theta_2,\cdots,\theta_{n-1}}(P), \cdots,$$
$$Q,\ R_{\theta_1,\theta_2,\cdots,\theta_{n-1}}(Q),\ R^2_{\theta_1,\theta_2,\cdots,\theta_{n-1}}(Q),\ \cdots,\ R^k_{\theta_1,\theta_2,\cdots,\theta_{n-1}}(Q), \cdots$$

are respectively non-Euclidean and Euclidean points.

In this case, if there exists an integer k such that $\theta_i | 2k\pi$ for all integers $1 \leq i \leq n-1$, then the previous sequences is finite. Thus there are both finite and infinite pseudo-Euclidean space $(\mathbf{R}^n, \mu)$ in this case. But if there is an integer i_0, $1 \leq i_0 \leq n-1$ such that $\theta_{i_0} \not| \ 2k\pi$ for any integer k, then there must be either infinite non-Euclidean points or infinite Euclidean points. Thus there are isometries of rotations in a finite non-Euclidean space only if there exists an integer k such that $\theta_i | 2k\pi$ for all integers $1 \leq i \leq n-1$. Similarly, an isometry of translation exists in this case only if $\theta_1, \theta_2, \cdots, \theta_{n-1} \in \{0, \pi/2\}$.

Reflection. By definition, a reflection F in a subspace E' of dimensional $n-1$ is an involution, i.e., $F^2 = 1_{\mathbf{R}^n}$. Thus if $(\mathbf{R}^n, \mu)$ is a pseudo-Euclidean space with an isometry of reflection F in E' and $P,\ Q \in (\mathbf{R}^n, \mu)$ are respectively a non-Euclidean point and a Euclidean point. Then it is only need that $P,\ F(P)$ are non-Euclidean points and $Q,\ F(Q)$ are Euclidean points. Therefore, a reflection F can be exists both in finite and infinite pseudo-Euclidean spaces $(\mathbf{R}^n, \mu)$.

Summing up all these discussions, we get results following for finite or infinite pseudo-Euclidean spaces.

Theorem 3.5 *Let $(\mathbf{R}^n, \mu)$ be a finite pseudo-Euclidean space. Then there maybe isometries of translations $T_{\overline{e}}$, rotations $R_{\overline{\theta}}$ and reflections on $(\mathbf{R}^n, \mu)$. Furthermore,*

(1) *If there are both isometries $T_{\overline{e}}$ and $R_{\overline{\theta}}$, where $\overline{e} = (e_1, \cdots, e_n)$ and $\overline{\theta} = (\theta_1, \cdots, \theta_{n-1})$, then $\theta_1, \theta_2, \cdots, \theta_{n-1} \in \{0, \pi/2\}$ and if $\theta_i = \pi/2$ for $1 \leq i \leq l$, $\theta_i = 0$ if $i \geq l+1$, then $e_1 = e_2 = \cdots = e_{l+1}$.*

(2) *If there is an isometry $R_{\theta_1,\theta_2,\cdots,\theta_{n-1}}$, then there must be an integer k such that $\theta_i \mid 2k\pi$ for all integers $1 \leq i \leq n-1$.*

(3) *There always exist isometries by putting Euclidean and non-Euclidean points* $\overline{x} \in \mathbf{R}^n$ *with* $\mu(\overline{x})$ *constant on symmetric positions to* E' *in* $(\mathbf{R}^n, \mu)$.

Theorem 3.6 *Let* $(\mathbf{R}^n, \mu)$ *be a infinite pseudo-Euclidean space. Then there maybe isometries of translations* $T_{\overline{e}}$, *rotations* $R_{\overline{\theta}}$ *and reflections on* $(\mathbf{R}^n, \mu)$. *Furthermore,*

(1) *There are both isometries* $T_{\overline{e}}$ *and* $R_{\overline{\theta}}$ *with* $\overline{e} = (e_1, e_2, \cdots, e_n)$ *and* $\overline{\theta} = (\theta_1, \theta_2, \cdots, \theta_{n-1})$, *only if* $\theta_1, \theta_2, \cdots, \theta_{n-1} \in \{0, \pi/2\}$ *and if* $\theta_i = \pi/2$ *for* $1 \leq i \leq l$, $\theta_i = 0$ *if* $i \geq l+1$, *then* $e_1 = e_2 = \cdots = e_{l+1}$.

(2) *There exist isometries of rotations and reflections by putting Euclidean and non-Euclidean points in the orbits* $\overline{x}^{\langle R_{\overline{\theta}} \rangle}$ *and* $\overline{y}^{\langle F \rangle}$ *with a constant* $\mu(\overline{x})$ *in* $(\mathbf{R}^n, \mu)$.

We determine isometries on $(\mathbf{R}^3, \mu)$ with a 3-cube $\mathcal{C}^3$ shown in Fig.9.4.2. Let $[\overline{a}]$ be an 3×3 orthogonal matrix, $[\overline{a}] \neq I_{3\times 3}$ and let $\mu(x_1, x_2, x_3) = [\overline{a}]$ for $x_1, x_2, x_3 \in \{0,\ 1\}$, otherwise, $\mu(x_1, x_2, x_3) = I_{3\times 3}$. Then its isometries consist of two types following:

Rotations:

$R_1,\ R_2,\ R_3$: these rotations through $\pi/2$ about 3 axes joining centres of opposite faces;

$R_4,\ R_5,\ R_6,\ R_7,\ R_8,\ R_9$: these rotations through π about 6 axes joining midpoints of opposite edges;

$R_{10},\ R_{11},\ R_{12},\ R_{13}$: these rotations through about 4 axes joining opposite vertices.

Reflection F: the reflection in the centre fixes each of the grand diagonal, reversing the orientations.

Then $\mathrm{Isom}(\mathbf{R}^3, \mu) = \langle R_i,\ F,\ 1 \leq i \leq 13 \rangle \simeq S_4 \times Z_2$. But if let $[\overline{b}]$ be another 3×3 orthogonal matrix, $[\overline{b}] \neq [\overline{a}]$ and define $\mu(x_1, x_2, x_3) = [\overline{a}]$ for $x_1 = 0,\ x_2, x_3 \in \{0,\ 1\}$, $\mu(x_1, x_2, x_3) = [\overline{b}]$ for $x_1 = 1,\ x_2, x_3 \in \{0,\ 1\}$ and $\mu(x_1, x_2, x_3) = I_{3\times 3}$ otherwise. Then only the rotations R, R^2, R^3, R^4 through $\pi/2,\ \pi,\ 3\pi/2$ and 2π about the axis joining centres of opposite face

$$\{(0,0,0), (0,0,1), (0,1,0), (0,1,1)\} \text{ and } \{(1,0,0), (1,0,1), (1,1,0), (1,1,1)\},$$

and reflection F through to the plane passing midpoints of edges

$$(0,0,0) - (0,0,1),\ (0,1,0) - (0,1,1),\ (1,0,0) - (1,0,1),\ (1,1,0) - (1,1,1)$$

or $$(0,0,0) - (0,1,0),\ (0,0,1) - (0,1,1),\ (1,0,0) - (1,1,0),\ (1,0,1) - (1,1,1)$$

are isometries on $(\mathbf{R}^3, \mu)$. Thus $\mathrm{Isom}(\mathbf{R}^3, \mu) = \langle R_1,\ R_2,\ R_3,\ R_4,\ F \rangle \simeq D_8$.

Furthermore, let $[\overline{a}_i]$, $1 \leq i \leq 8$ be orthogonal matrixes distinct two by two and define $\mu(0,0,0) = [\overline{a}_1]$, $\mu(0,0,1) = [\overline{a}_2]$, $\mu(0,1,0) = [\overline{a}_3]$, $\mu(0,1,1) = [\overline{a}_4]$, $\mu(1,0,0) = [\overline{a}_5]$, $\mu(1,0,1) = [\overline{a}_6]$, $\mu(1,1,0) = [\overline{a}_7]$, $\mu(1,1,1) = [\overline{a}_8]$ and $\mu(x_1, x_2, x_3) = I_{3\times 3}$ if $x_1,\ x_2,\ x_3 \neq 0$ or 1. Then $\mathrm{Isom}(\mathbf{R}^3, \mu)$ is nothing but a trivial group.

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On Fuzzy Matroids

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Abstract: The aim of this paper is to discuss properties of fuzzy regular-flats, fuzzy C-flats, fuzzy alternative-sets and fuzzy i-flats. Moreover, we characterize some peculiar fuzzy matroids via these notions. Finally, we provide a decomposition of fuzzy strong maps.

Key Words: Neutrosophic set, fuzzy matroid, fuzzy flat, fuzzy closure, fuzzy strong map, fuzzy hesitant map.

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§1. Introduction

The matroid theory has several interesting applications in system analysis, operations research and economics. Since most of the time the aspects of matroid problems are uncertain, it is nice to deal with these aspects via the methods of fuzzy logic. The notion of fuzzy matroids was first introduced by Geotschel and Voxman in their landmark paper [4] using the notion of fuzzy independent set. The notion of fuzzy independent set was also explored in [10,9]. Some constructions, fuzzy spanning sets, fuzzy rank and fuzzy closure axioms were also studied in [5-7,13]. Several other fuzzifications of matroids were also discussed in [8,11]. Since the notion of flats in traditional matroids is one of the most significant notions that plays a very important rule in characterizing strong maps (see for example [3,12]). In [2], the notions of fuzzy flats and fuzzy closure flats were introduced and several examples were provided. Thus in [2], fuzzy matroids are defined via fuzzy flats axioms and it was shown that the levels of the fuzzy matroid introduced are indeed crisp matroids. Moreover, fuzzy strong maps and fuzzy hesitant maps are introduced and explored. We remark that the approach in [2] is different from those mentioned above. Let $FM = (E,\mathcal{O})$ be a fuzzy matroid. A fuzzy set $\lambda \in E$ is called a *fuzzy C-open set* in FM if there exists a fuzzy open set μ such that $\mu \leq \lambda \leq \bar{\mu}$ ([1]).

Let E be any non-empty set. A *neutrosophic set* based on neutrosophy, is defined for an element $x(T, I, F)$ belongs to the set if it is t true in the set, i indeterminate in the set, and f false, where t, i and f are real numbers taken from the sets T, I and F with no restriction on T, I, F nor on their sum $n = t + i + f$. Particularly, if $I = \emptyset$, we get the fuzzy set. By $\wp(1)$ we denote the set of all fuzzy sets on E. That is $\wp(1) = [0, 1]^E$, which is a completely distributive lattice. Thus let 0^E and 1^E denote its greatest and smallest elements, respectively. That is $0^E(e) = 0$ and $1^E(e) = 1$ for every $e \in E$. A fuzzy set μ_1 is a subset of μ_2 , written $\mu_1 \leq \mu_2$, if

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$\mu_1(e) \leq \mu_2(e)$ for all $e \in E$. If $\mu_1 \leq \mu_2$ and $\mu_1 \neq \mu_2$, then μ_1 is a proper subset of μ_2, written $\mu_1 < \mu_2$. Moreover, $\mu_1 \prec \mu_2$ if $\mu_1 < \mu_2$ and there does not exist μ_3 such that $\mu_1 < \mu_3 < \mu_2$. Finally, $\mu_1 \vee \mu_2 = \sup\{\mu_1, \mu_2\}$ and $\mu_1 \wedge \mu_2 = \inf\{\mu_1, \mu_2\}$.

Next we recall some basic definitions and results from [2].

Definition 1.1 *Let E be a finite set and let $\mathfrak{F}$ be a family of fuzzy subsets of E satisfying the following three conditions:*

(*i*) $1^E \in \mathfrak{F}$;

(*ii*) *If $\mu_1, \mu_2 \in \mathfrak{F}$, then $\mu_1 \wedge \mu_2 \in \mathfrak{F}$;*

(*iii*) *If $\mu \in \mathfrak{F}$ and $\mu_1, \mu_2, ..., \mu_n$ are all minimal members of $\mathfrak{F}$ (with respect to standard fuzzy inclusion) that properly contain μ (in this case we write $\mu \prec \mu_i$ for all $i = 1, 2, ..., n$), then the fuzzy union of $\mu_1, \mu_2, ..., \mu_n$ is equal to 1^E (i.e. $\vee_{i=1}^n \mu_i = 1^E$). Then the system $FM = (E, \mathfrak{F})$ is called fuzzy matroid and the elements of $\mathfrak{F}$ are fuzzy flats of FM.*

Definition 1.2 *For $r \in (0, 1]$, let $C^r(\mu) = \{e \in E | \mu(e) \geq r\}$ be the r-level of a fuzzy set $\mu \in \mathfrak{F}$, and let $\mathfrak{F}^r = \{C^r(\mu) : \mu \in \mathfrak{F}\}$ be the r-level of the family $\mathfrak{F}$ of fuzzy flats. Then for $r \in (0, 1]$, $(E, \mathfrak{F}^r)$ is the* r-level *of the fuzzy set system $(E, \mathfrak{F})$.*

Theorem 1.3 *For every $r \in (0, 1]$, $\mathfrak{F}^r = \{C^r(\mu) : \mu \in \mathfrak{F}\}$ the r-levels of a family of fuzzy flats $\mathfrak{F}$ of a fuzzy matroid $FM = (E, \mathfrak{F})$ is a family of crisp flats.*

Definition 1.4 *Let E be any set with n-elements and $\mathfrak{F} = \{\chi_A : A \leq E, \ |A| = n \text{ or } |A| < m\}$ where m is a positive integer such that $m \leq n$. Then $(E, \mathfrak{F})$ is a fuzzy matroid called the fuzzy uniform matroid on n-elements and rank m, denoted by $F_{m,n}$. $F_{m,m}$ is called the free fuzzy uniform matroid on n-elements.*

We remark that the rank notion in the preceding definition coincides with that in [6].

Definition 1.5 *Let $FM = (E, \mathfrak{F})$ be a fuzzy matroid and $\mu \in \mathfrak{F}$. Then the fuzzy closure of μ is $\bar{\mu} = \bigwedge_{\lambda \in \mathfrak{F}, \mu \leq \lambda} \lambda$.*

Theorem 1.6 *Let $FM = (E, \mathfrak{F})$ be a fuzzy matroid and X be a non-empty subset of E. Then $(X, \mathfrak{F}_X)$ is a fuzzy matroid, where $\mathfrak{F}_X = \{\chi_X \wedge \mu : \mu \in \mathfrak{F}\}$.*

Let $FM = (E, \mathfrak{F})$ be a fuzzy matroid, X be a non-empty subset of E and μ be a fuzzy set in X. We may realize μ as a fuzzy set in E by the convention that $\mu(e) = 0$ for all $e \in E - X$. It can be easily shown that $\mathfrak{F}_X = \{\mu|_X : \mu \in \mathfrak{F}\}$, where $\mu|_X$ is the restriction of μ to X.

Let E_1 and E_2 be two sets, μ_1 is a fuzzy set in E_1, μ_2 is a fuzzy set in E_2 and $f : E_1 \to E_2$ be a map. Then we define the fuzzy sets $f(\mu_1)$ (the image of μ_1) and $f^{-1}(\mu_2)$ (the preimage of μ_2) by

$$f(\mu_1)(y) = \begin{cases} \sup\{\mu_1(x) : x \in f^{-1}(\{y\})\} & , y \in \text{Range}(f) \\ 1 & , \text{Otherwise}, \end{cases}$$

and $f^{-1}(\mu_2)(x) = \mu_2(f(x))$ for all $x \in E_1$.

Definition 1.7 *A fuzzy strong map from a fuzzy matroid $FM_1 = (E_1, \mathfrak{F}_1)$ into a fuzzy matroid $FM_2 = (E_2, \mathfrak{F}_2)$ is a map $f : E_1 \to E_2$ such that the preimage of every fuzzy flat in FM_2 is a fuzzy flat in FM_1.*

Theorem 1.8 *Let $FM_1 = (E_1, \mathfrak{F}_1)$ and $FM_2 = (E_2, \mathfrak{F}_2)$ be fuzzy matroids and $f : E_1 \to E_2$ be a map. Then the following are equivalent:*

- (i) *f is fuzzy strong;*
- (ii) *For every fuzzy set μ_1 in FM_1, $f(\overline{\mu_1}) \leq \overline{f(\mu_1)}$;*
- (iii) *For every fuzzy set μ_2 in FM_2, $\overline{f^{-1}(\mu_2)} \leq f^{-1}(\overline{\mu_2})$.*

Next, we recall some results from [1].

Definition 1.9 *Let $FM = (E, \mathfrak{F})$ be a fuzzy matroid and μ be a fuzzy set. Then μ is a fuzzy c-flat if $\bigvee_{\lambda \in \mathfrak{F}, \lambda \leq \overline{\mu}} \lambda \leq \mu$.*

Clearly, every fuzzy flat is a fuzzy c-flat, but the converse need not be true.

Example 1.1 Let $E = \{a, b, c, d\}$ and $\mathfrak{F} = \{1^E, 0, \chi_{\{a,b\}}, \chi_{\{c,d\}}\}$. $FM = (E, \mathfrak{F})$ is a fuzzy matroid. $\chi_{\{b,d\}}$ is a fuzzy c- flat that is not a fuzzy flat.

Lemma 1.10 *The intersection of fuzzy c-flats is a fuzzy c-flat.*

Lemma 1.11 *Let $FM = (E, \mathfrak{F})$ be a fuzzy matroid and μ be a fuzzy set. The fuzzy C closure of μ is $\overline{\mu}^F = \bigwedge\{\acute{\mu} : \acute{\mu}$ is a fuzzy c-flat and $\mu \leq \acute{\mu}\}$.*

Theorem 1.12 *Let $FM = (E, \mathfrak{F})$ be a fuzzy matroid and μ, λ be fuzzy sets. Then*

- *i)* $\overline{0}^F = 0$;
- *ii)* $\overline{\mu}^F$ *is a fuzzy closure flat;*
- *iii)* $\mu \leq \overline{\mu}^F$;
- *iv)* *If $\mu \leq \lambda$, then $\overline{\mu}^F \leq \overline{\lambda}^F$;*
- *v)* $\overline{\overline{\mu}^F}^F = \overline{\mu}^F$.

Lemma 1.13 *Let $FM = (E, \mathfrak{F})$ be a fuzzy matroid and μ be a fuzzy set. Then μ is a fuzzy c-flat if and only if $\overline{\mu}^F = \mu$.*

Lemma 1.14 *Let $FM = (E, \mathfrak{F})$ be a fuzzy matroid and μ, λ be fuzzy sets. Then*

- *i)* $\overline{\mu \vee \lambda}^F \geq \overline{\mu}^F \vee \overline{\lambda}^F$;
- *ii)* $\overline{\mu \wedge \lambda}^F \leq \overline{\mu}^F \wedge \overline{\lambda}^F$.

Definition 1.15 *A map $f : FM_1 \to FM_2$ is*

- *i) fuzzy c-strong if the inverse image of every fuzzy flat of FM_2 is a fuzzy c-flat of FM_1;*
- *ii) fuzzy hesitant if the inverse image of every fuzzy c-flat of FM_2 is a fuzzy c-flat of FM_1.*

Clearly, a fuzzy strong (fuzzy hesitant) map is fuzzy c-strong, but the converse need not be true since a fuzzy c-flat need not be a fuzzy flat as we have seen in Example 1.1.

A map $f : FM_1 \to FM_2$ is said to be *fuzzy* if the image of every fuzzy flat of FM_1 is a fuzzy flat of FM_2. The following is a trivial result.

Lemma 1.16 *Let $f : FM_1 \to FM_2$ be a fuzzy map that is also fuzzy strong. Then $f^{-1}(\bar{\mu}) = \overline{f^{-1}(\mu)}$ for every fuzzy set μ of FM_2.*

Theorem 1.17 *A fuzzy map $f : FM_1 \to FM_2$ that is also fuzzy strong is fuzzy hesitant.*

Theorem 1.18 *The following are equivalent for a map $f : FM_1 \to FM_2$:*

i) f is hesitant;
ii) $f(\overline{\mu}^F) \le \overline{f(\mu)}^F$ for every fuzzy set μ of FM_1;
iii) $\overline{f^{-1}(\lambda)}^F \le f^{-1}(\overline{\lambda}^F)$ for every fuzzy set λ of FM_2.

§2. Fuzzy-Regular- and Fuzzy i-Flats

In this section, the notions of fuzzy–regular-flat and fuzzy-i-flats are discussed. We prove that the notion of fuzzy-i-flat coincides with that of fuzzy-c-flat. In addition, we provide several characterizations of fuzzy-regular-flats and fuzzy open sets of certain fuzzy matroids.

Definition 2.1 *Let $FM = (E,\mathcal{O})$ be a fuzzy matroid. A fuzzy set λ is* nowhere-spanning boundary *if $\overset{o}{\overline{\bar{\lambda}\backslash\overset{o}{\lambda}}} = 0$,* fuzzy local-flat *if $\lambda = \mu \wedge \bar{\lambda}$, where μ is fuzzy open and λ is fuzzy C-preopen if $\mu \le \overset{o}{\bar{\mu}}$ and* fuzzy-i-flat *if $\overset{o}{\bar{\lambda}} = \overset{o}{\lambda}$.*

The following example shows that a nowhere-spanning-boundary-fuzzy set needs not be a fuzzy-c-flat. In the next coming theorem we prove a partial converse of this.

Example 2.1 Let $E = \{a, b, c\}$ and $\mathcal{O} = \{1, \chi_{\{b,c\}}\}$. Then $\chi_{\{a,b\}}$ is an nowhere-spanning-boundary-fuzzy set but not a fuzzy-c-flat.

Theorem 2.2 *In a loopless fuzzy matroid, every fuzzy-c-flat is a nowhere-spanning-boundary-fuzzy set.*

Proof Clearly the intersection of two nowhere-spanning-boundary-fuzzy sets is a nowhere-spanning-boundary-fuzzy set. Since a fuzzy-c-flat is an intersection of a (fuzzy C-open) fuzzy open set and a fuzzy closure flat, it is enough to show that every fuzzy C-open and every fuzzy c-flat is a nowhere-spanning-boundary-fuzzy set. If λ is a C-open, then for some fuzzy open set μ we have $\mu \le \lambda \le \bar{\mu}$. Since $\bar{\lambda}\backslash\overset{o}{\lambda} \le \bar{\mu}\backslash\overset{o}{\mu} = 0$. Thus $\overset{o}{\overline{\bar{\lambda}\backslash\overset{o}{\lambda}}} = 0$. In fact, it is obvious that every fuzzy open set is a nowhere-spanning-boundary-fuzzy set. Thus fuzzy C–open- (and hence every fuzzy C–flat-) set is a nowhere-spanning-boundary-fuzzy set. □

The following result shows that the fuzzy-i-flats class coincides with the class of closure flats.

Theorem 2.3 *Let $M = (E, \mathcal{O})$ be a loopless fuzzy matroid. Then the following are equivalent:*

(1) *λ is a fuzzy-i-flat;*

(2) *λ is a closure flat;*

(3) *$1 - \lambda$ is a C-preopen and λ is a fuzzy-c-flat;*

(4) *$1 - \lambda$ is a C-preopen and λ is a nowhere-spanning-boundary-set.*

Proof (1) $\Rightarrow$ (2) Since $\overset{o}{\bar{\lambda}} = \overset{o}{\lambda} \leq \lambda$, then $1 - \lambda \leq \overline{(1 - \lambda)^o}$. Thus $1 - \lambda$ is fuzzy C-open, hence isλ a fuzzy-c-flat.

(2) $\Rightarrow$ (3) Every fuzzy-c-flat is trivially fuzzy C-preopen. Since $\lambda = 1 \wedge \lambda$, where 1 is fuzzy open and λ is a fuzzy-c-flat, then $1 - \lambda$ is an fuzzy-c-flat.

(3) $\Rightarrow$ (4) Theorem 2.2.

(4) $\Rightarrow$ (1) Since λ is a nowhere-spanning-boundary-fuzzy set, $\mu = 1 - \lambda$ is also a nowhere-spanning-boundary-fuzzy set and as

$$(\bar{\mu} \backslash \overset{o}{\backslash \mu})^o = \overset{o}{\bar{\mu}} \wedge \overset{o}{\overline{1 - \mu}} = \overset{o}{\bar{\mu}} \wedge (1 - \overline{\overset{o}{\bar{\mu}}}) = \overset{o}{\bar{\mu}} \backslash \overline{\overset{o}{\bar{\mu}}},$$

it follows that $\overset{o}{\bar{\mu}} \leq \overline{\overset{o}{\bar{\mu}}}$. Since μ is fuzzy C-preopen, $\mu \leq \overline{\overset{o}{\bar{\mu}}}$. Thus $\mu \leq \overline{\overset{o}{\mu}}$ or equivalently $\bar{\mu} = \overline{\overset{o}{\mu}}$. Since $\mu = 1 - \lambda$, $\overset{o}{\bar{\lambda}} = \overset{o}{\lambda}$. $\square$

A matroid $M = (E, \mathcal{O})$ is called *a fuzzy closure matroid* if $\overline{\lambda \vee \mu} = \bar{\lambda} \vee \bar{\mu}$ for all fuzzy subsets λ and μ of E. Next we characterize the class of fuzzy-regular-flats of a fuzzy closure matroid. We show that the class of fuzzy-regular-flats is the intersection of the class of fuzzy local-flats with either the class of fuzzy C-open-sets or the class of fuzzy C-preopen-sets.

Theorem 2.4 *Let $M = (E, \mathcal{O})$ be a fuzzy closure matroid and $\lambda \in E$. Then the following are equivalent:*

(1) *λ is a fuzzy-regular-flat;*

(2) *λ is fuzzy C–open set and a fuzzy local-flat;*

(3) *λ is fuzzy C-preopen-set and a fuzzy local-flat.*

Proof (1) $\Rightarrow$ (2) Every fuzzy-regular-flat is clearly a fuzzy local-flat. Let $\lambda = \mu \wedge \eta$ be an fuzzy-regular-flat, where μ is fuzzy open and η is a fuzzy flat such that $\eta = \overline{\overset{o}{\eta}}$. Since $\lambda = \mu \wedge \eta$, we have $\mu \wedge \eta^o \leq \overset{o}{\lambda}$. It is easily seen that $\overset{o}{\lambda} \leq \lambda \leq \eta$, hence $\overset{o}{\lambda} \leq \eta^o$. But $\overset{o}{\lambda} \leq \lambda \leq \mu$, hence $\overset{o}{\lambda} \leq \mu \wedge \eta^o$. Therefore $\overset{o}{\lambda} = \mu \wedge \eta^o$. Now we prove $\lambda \leq \overline{\lambda^o}$. Let $e \in \lambda$ and δ be an fuzzy open set containing e. Then $e \in \mu \wedge \delta = (1 - \lambda_1) \wedge (1 - \lambda_2)$ for some fuzzy flats λ_1 and λ_2. Thus $e \in 1 - (\lambda_1 \vee \lambda_2) = 1 - (\overline{\lambda_1} \vee \overline{\lambda_2}) = 1 - (\overline{\lambda_1 \vee \lambda_2})$ which is fuzzy open. Since $e \in \eta = \overline{\overset{o}{\eta}}$, there exists $l \in \overset{o}{\eta}$ such that $l \neq e$ and $l \in 1 - (\overline{\lambda_1 \vee \lambda_2}) = \mu \wedge \eta$. Hence $l \in \mu \wedge \eta^o = \overset{o}{\lambda}$. Therefore $e \in \overline{\overset{o}{\lambda}}$ and $\lambda \leq \overline{\overset{o}{\lambda}}$. From $\overset{o}{\lambda} \leq \lambda \leq \overline{\overset{o}{\lambda}}$ we know that λ is fuzzy C-open.

(2) $\Rightarrow$ (3) is trivial

(3) $\Rightarrow$ (1) Since λ is a fuzzy local-flat, $\lambda = \mu \wedge \bar{\lambda}$, where μ is fuzzy open. As λ is fuzzy C-preopen and as $\overline{\overset{o}{\bar{\lambda}}} \leq \bar{\lambda}$, $\bar{\lambda}$ is a fuzzy regular-flat. Thus λ is an fuzzy-regular-flat. $\square$

Next, we characterize the class of fuzzy open sets of a loopless closure fuzzy matroid, hence we characterize the class of fuzzy flats.

Theorem 2.5 *Let $M = (E, \mathcal{O})$ be a loopless fuzzy closure matroid and $\lambda \in E$. Then the following are equivalent:*

(1) *λ is fuzzy open;*

(2) *λ is fuzzy prespanning and a fuzzy local-flat;*

(3) *λ is fuzzy prespanning and an fuzzy-regular-flat;*

(4) *λ is fuzzy prespanning and an fuzzy-c-flat.*

Proof (1) $\Rightarrow$ (2) Since $\lambda \leq \bar{\lambda}$, $\lambda = \overset{o}{\lambda} \leq \overset{o}{\bar{\lambda}}$. Thus λ is fuzzy prespanning. As 1 is a fuzzy flat and $\lambda = \overset{o}{\lambda} \wedge 1$, λ is a fuzzy local-flat.

(2) $\Rightarrow$ (3) Since λ is a fuzzy local-flat, $\lambda = \mu \wedge \bar{\lambda}$, where μ is fuzzy open. Since $\overset{o}{\bar{\lambda}} \leq \bar{\lambda}$, $\overline{\overset{o}{\bar{\lambda}}} \leq \bar{\lambda}$. But as λ is fuzzy prespanning, $\lambda \leq \overset{o}{\bar{\lambda}}$ and thus $\bar{\lambda} \leq \overline{\overset{o}{\bar{\lambda}}}$. Hence $\bar{\lambda}$ is a fuzzy regular-flat and so λ is a fuzzy-regular-flat.

(3) $\Rightarrow$ (4) Clearly a fuzzy flat is a fuzzy-i-flat and thus a fuzzy-regular-flat is an fuzzy-c-flat.

(4) $\Rightarrow$ (1) Since λ is an fuzzy-c-flat, we have $\lambda = \mu \wedge \eta$ where μ is open and $\overset{o}{\bar{\eta}} = \overset{o}{\eta}$. Because λ is fuzzy prespanning, we have $\lambda \leq \overset{o}{\bar{\lambda}} = \overset{o}{\overline{(\mu \wedge \eta)}} \subseteq \overset{o}{\bar{\mu}} \wedge \overset{o}{\bar{\eta}}$. Hence $\lambda = (\mu \wedge \eta) \wedge \mu \leq \mu \wedge \overset{o}{\eta}$. Notice $\lambda = \mu \wedge \eta \geq \mu \wedge \overset{o}{\eta}$, we have $\lambda = \mu \wedge \overset{o}{\eta}$. Thus as M is a closure fuzzy matroid, λ is fuzzy open.$\square$

§3. Characterizations of Particular Fuzzy Matroids

In this section, we characterize maximal fuzzy matroids, local-flat-fuzzy matroids, free fuzzy matroids and others via fuzzy-regular-flats and fuzzy-c-flats. We provide a decomposition of fuzzy strong maps at the end of this section.

Theorem 3.1 *For a loopless fuzzy matroid $M = (E, \mathcal{O})$, the following are equivalent:*

(1) *M is maximal;*

(2) *Every fuzzy subset of E is an fuzzy-c-flat;*

(3) *Every spanning fuzzy subset of E is an fuzzy-c-flat.*

Proof (1) $\Rightarrow$ (2) Let $\lambda \in E$. Since every submatroid of a maximal fuzzy matroid is maximal, then $M|\bar{\lambda}$ is maximal. Since $\underline{\lambda}$ is a spanning fuzzy set in $M|\bar{\lambda}$, λ is fuzzy open in $M|\bar{\lambda}$. Thus $\lambda = \mu \wedge \bar{\lambda}$ where μ is a fuzzy open set in M and $\bar{\lambda}$ is a fuzzy-c-flat in M. Hence λ is an fuzzy-c-flat.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) Let $\bar{\lambda} = 1$. By (3) $\lambda = \mu \wedge \eta$, where μ is fuzzy open and η is a fuzzy-c-flat. Since $\lambda \leq \eta$, $\bar{\eta} = 1$ and hence $\overset{o}{\eta} = \overset{o}{\bar{\eta}} = \overset{o}{1} = 1$, since M is loopless. Thus $\eta = 1$ and $\lambda = \mu$ is fuzzy open. Therefore, M is maximal. $\square$

Theorem 3.2 *Let $M = (E, \mathcal{O})$ be a loopless fuzzy closure matroid. Then the following are*

equivalent:

(1) *M is a locally-couniform fuzzy matroid;*
(2) *Every fuzzy-c-flat is both fuzzy open and a fuzzy flat;*
(3) *Every fuzzy-c-flat is a fuzzy flat.*

Proof (1) $\Rightarrow$ (2) If λ is an fuzzy-c-flat, $\lambda = \mu \wedge \eta$, where μ is fuzzy open and η is a C-fuzzy flat. By (1) μ is also a fuzzy flat. On the other hand η is fuzzy open by (1) and thus $\overset{o}{\overline{\eta}} \leq \eta \leq \eta$ and so η is both fuzzy open and a fuzzy flat. Therefore, λ is a fuzzy flat being the intersection of two fuzzy flats and as M is a closure fuzzy matroid, λ is also fuzzy open.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) Every fuzzy open set is a fuzzy-c-flat by Theorem 2.5 and thus by (3) a fuzzy flat. □

Theorem 3.3 *Let $M = (E, \mathcal{O})$ be a loopless fuzzy closure matroid. Then the following are equivalent:*

(1) *$M \cong F_{1,n}$ for some positive integer $n \geq 1$;*
(2) *The only fuzzy-c-flats in M are the trivial ones;*
(3) *The only fuzzy-regular-flats in M are the trivial ones.*

Proof (1) $\Rightarrow$ (2) If λ is an fuzzy-c-flat, then $\lambda = \mu \wedge \eta$, where μ is fuzzy open and η is a fuzzy-c-flat ($\eta^o = \overline{\eta^o}$). If $\lambda \neq 0$, then $\mu \neq 0$ and by (1) $\mu = 1$. Thus $\lambda = \eta$ and so $\lambda^o = \overset{o}{\bar{\lambda}} = 1^o = 1$. Hence $\lambda = 1$.

(2) $\Rightarrow$ (3) Every fuzzy-regular-flat is an fuzzy-c-flat.

(3) $\Rightarrow$ (1) Since every fuzzy open set is a fuzzy-regular-flat, by (3) the only fuzzy open sets in M are the trivial ones. □

It is well-known that the notions of fuzzy-regular-flat and fuzzy C-open-set are independent from each other. By Theorem 2.4 in a fuzzy closure matroid, every fuzzy-regular-flat is fuzzy C-open. Clearly a fuzzy-regular-flat is a fuzzy local-flat in any fuzzy matroid. Next we show that a fuzzy C-open-set which is also a fuzzy local-flat has to be a fuzzy-regular-flat.

Theorem 3.4 *In any fuzzy matroid, every fuzzy set λ that is both fuzzy C-open and a fuzzy local-flat is a fuzzy-regular-flat.*

Proof Since λ is fuzzy C-open, $\lambda \leq \overline{\overset{o}{\lambda}}$ and since λ is a fuzzy local-flat, $\lambda = \mu \wedge \bar{\lambda}$, where μ is fuzzy open. Then $\bar{\lambda} = \overline{\overset{o}{\lambda}}$ and so $\overline{\lambda}$ is a fuzzy regular-flat. Hence λ is a fuzzy-regular-flat. □

Corollary 3.5 *Let $M = (E, \mu)$ be a fuzzy closure matroid and $\lambda \leq 1$. Then λ is a fuzzy-regular-flat if and only if λ is both fuzzy C-open-set and a fuzzy local-flat.*

Theorem 3.6 *Let $M = (E, \mu)$ be a loopless fuzzy closure matroid. Then M is free if and only if every fuzzy subset of E is a fuzzy-regular-flat.*

Proof Let M be free. Then every fuzzy set $\lambda \leq 1$ is open and a fuzzy regular-flat. Hence

λ is a fuzzy-regular-flat.

Conversely if every fuzzy subset of E is a fuzzy-regular-flat, then every singleton $e \leq 1$ is a fuzzy-regular-flat and by Theorem 2.4 fuzzy C-open. If $e^o = 0$, then we have the contradiction $e \leq \overline{e^o} = 0$. Thus $e = e^o$ or equivalently every singleton is fuzzy open. Thus every fuzzy subset of E is fuzzy open and hence M is free. □

Theorem 3.7 *Let $M = (E, \mathcal{O})$ be a loopless fuzzy closure matroid and $\lambda \leq 1$. Then the following are equivalent:*

(1) *λ is fuzzy open;*

(2) *λ is a alternative-fuzzy set and a fuzzy local-flat;*

(3) *λ is fuzzy prespanning and a fuzzy local-flat.*

Proof (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are trivial.

(3) $\Rightarrow$ (1) Let λ be a fuzzy prespanning set that is also a fuzzy local-flat. Then $\lambda \leq \overset{o}{\bar{\lambda}}$ and $\lambda = \mu \wedge \bar{\lambda}$, where μ is fuzzy open. Thus $\lambda \leq \mu \wedge \overset{o}{\bar{\lambda}} = (\mu \wedge \bar{A})^o = \overset{o}{\lambda}$. Therefore, λ is fuzzy open.□

Definition 3.8 *A fuzzy map $f : M_1 \rightarrow M_2$ is called* fuzzy hesitant *(resp.* fuzzy alternative-strong, fuzzy prestrong, fuzzy local–flat-strong, fuzzy open–regular-flat-strong*) if the inverse image of every open set in M_2 is a fuzzy C-open (resp. fuzzy alternative-set, fuzzy prespanning set, fuzzy local-flat, fuzzy-regular-flat) in M_1.*

Combining Corollary 3.5 and Theorem 3.4, we immediately obtain the following decomposition of fuzzy strong maps.

Theorem 3.9 *Let $f : M_1 \rightarrow M_2$ be a fuzzy map where M_1 is a loopless fuzzy closure matroid. Then*

(*i*) *f is fuzzy open–regular-flat-strong if and only if f is fuzzy hesitant and fuzzy local-flat-strong;*

(*ii*) *f is fuzzy strong if and only if f is fuzzy alternative-strong and fuzzy local-flat-strong;*

(*iii*) *f is fuzzy alternative-strong if and only if f is fuzzy prestrong and fuzzy hesitant;*

(*iv*) *f is fuzzy strong if and only if f is fuzzy prestrong and fuzzy local-flat-strong;*

(*v*) *f is fuzzy strong if and only if f is fuzzy prestrong and fuzzy open–regular-flat-strong.*

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Matrix Representation of Biharmonic Curves in Terms of Exponential Maps in the Special Three-Dimensional $\phi-$Ricci Symmetric Para-Sasakian Manifold

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Abstract: In this paper, we study biharmonic curves in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$. We construct matrix representation of biharmonic curves in terms of exponential maps in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$.

Key Words: Biharmonic curve, bienergy, bitension field, para-Sasakian manifold, exponential map.

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§1. Introduction

In the theory of Lie groups the exponential map is a map from the Lie algebra of a Lie group to the group which allows one to recapture the local group structure from the Lie algebra. The existence of the exponential map is one of the primary justifications for the study of Lie groups at the level of Lie algebras.

The ordinary exponential function of mathematical analysis is a special case of the exponential map when G is the multiplicative group of non-zero real numbers (whose Lie algebra is the additive group of all real numbers). The exponential map of a Lie group satisfies many properties analogous to those of the ordinary exponential function, however, it also differs in many important respects.

The aim of this paper is to study matrix representation of exponential maps in terms of biharmonic curves in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$.

A smooth map $\phi : N \longrightarrow M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} |\mathcal{T}(\phi)|^2 \, dv_h,$$

where $\mathcal{T}(\phi) := \mathrm{tr}\nabla^{\phi} d\phi$ is the tension field of ϕ

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The Euler–Lagrange equation of the bienergy is given by $\mathcal{T}_2(\phi)=0$. Here the section $\mathcal{T}_2(\phi)$ is defined by

$$\mathcal{T}_2(\phi)=-\Delta_\phi\mathcal{T}(\phi)+\mathrm{tr}R\left(\mathcal{T}(\phi),d\phi\right)d\phi, \tag{1.1}$$

and called the bitension field of ϕ. Non-harmonic biharmonic maps are called proper biharmonic maps.

In this paper, we study biharmonic curves in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$. We construct matrix representation of biharmonic curves in terms of exponential maps in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$.

§2. Preliminaries

An n-dimensional differentiable manifold M is said to admit an almost para-contact Riemannian structure (ϕ,ξ,η,g), where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is a Riemannian metric on M such that

$$\phi\xi=0,\quad \eta\left(\xi\right)=1,g\left(X,\xi\right)=\eta\left(X\right), \tag{2.1}$$

$$\phi^2\left(X\right)=X-\ \eta\left(X\right)\xi, \tag{2.2}$$

$$g\left(\phi X,\phi Y\right)=g\left(X,Y\right)-\eta\left(X\right)\eta\left(Y\right), \tag{2.3}$$

for any vector fields X, Y on M.

In addition, if (ϕ,ξ,η,g), satisfy the equations

$$d\eta=0,\quad \nabla_X\xi=\phi X, \tag{2.4}$$

$$\left(\nabla_X\phi\right)Y=-g\left(X,Y\right)\xi-\eta\left(Y\right)X+2\eta\left(X\right)\eta\left(Y\right)\xi,\qquad X,Y\in\chi\left(M\right), \tag{2.5}$$

then M is called a para-Sasakian manifold or, briefly a $P-$Sasakian manifold [2].

Definition 2.1 *A para-Sasakian manifold M is said to be locally ϕ-symmetric if*

$$\phi^2\left(\left(\nabla_WR\right)\left(X,Y\right)Z\right)=0,$$

for all vector fields X,Y,Z,W orthogonal to ξ. This notion was introduced by Takahashi [16], for a Sasakian manifold.

Definition 2.2 *A para-Sasakian manifold M is said to be ϕ-symmetric if*

$$\phi^2\left(\left(\nabla_WR\right)\left(X,Y\right)Z\right)=0,$$

for all vector fields X,Y,Z,W on M.

Definition 2.3 *A para-Sasakian manifold M is said to be ϕ-Ricci symmetric if the Ricci operator satisfies*

$$\phi^2\left(\left(\nabla_XQ\right)\left(Y\right)\right)=0,$$

for all vector fields X and Y on M and $S(X,Y)=g(QX,Y)$.

If X,Y are orthogonal to ξ, then the manifold is said to be locally ϕ-Ricci symmetric.

We consider the three-dimensional manifold

$$\mathbb{P}=\left\{\left(x^1,x^2,x^3\right)\in\mathbb{R}^3:\left(x^1,x^2,x^3\right)\neq(0,0,0)\right\},$$

where $\left(x^1,x^2,x^3\right)$ are the standard coordinates in $\mathbb{R}^3$. We choose the vector fields

$$\mathbf{e}_1=e^{x^1}\frac{\partial}{\partial x^2},\quad \mathbf{e}_2=e^{x^1}\left(\frac{\partial}{\partial x^2}-\frac{\partial}{\partial x^3}\right),\quad \mathbf{e}_3=-\frac{\partial}{\partial x^1} \tag{2.6}$$

are linearly independent at each point of $\mathbb{P}$.

Let η be the 1-form defined by

$$\eta(Z)=g(Z,\mathbf{e}_3)\text{ for any }Z\in\chi(\mathbb{P}).$$

Let be the (1,1) tensor field defined by

$$\phi(\mathbf{e}_1)=\mathbf{e}_2,\ \phi(\mathbf{e}_2)=\mathbf{e}_1,\ \phi(\mathbf{e}_3)=0.$$

Then using the linearity of and g we have

$$\eta(\mathbf{e}_3)=1,$$

$$\phi^2(Z)=Z-\ \eta(Z)\mathbf{e}_3,$$

$$g\left(\phi Z,\phi W\right)=g\left(Z,W\right)-\eta(Z)\eta(W),$$

for any $Z,W\in\chi(\mathbb{P})$. Thus for $\mathbf{e}_3=\xi$, (ϕ,ξ,η,g) defines an almost para-contact metric structure on $\mathbb{P}$.

Let ∇ be the Levi-Civita connection with respect to g. Then, we have

$$[\mathbf{e}_1,\mathbf{e}_2]=0,\quad [\mathbf{e}_1,\mathbf{e}_3]=\mathbf{e}_1,\quad [\mathbf{e}_2,\mathbf{e}_3]=\mathbf{e}_2.$$

Taking $\mathbf{e}_3=\xi$ and using the Koszul's formula, we obtain

$$\begin{aligned}\nabla_{\mathbf{e}_1}\mathbf{e}_1&=-\mathbf{e}_3,\ \nabla_{\mathbf{e}_1}\mathbf{e}_2=0,\qquad \nabla_{\mathbf{e}_1}\mathbf{e}_3=\mathbf{e}_1,\\ \nabla_{\mathbf{e}_2}\mathbf{e}_1&=0,\qquad \nabla_{\mathbf{e}_2}\mathbf{e}_2=-\mathbf{e}_3,\ \nabla_{\mathbf{e}_2}\mathbf{e}_3=\mathbf{e}_2,\\ \nabla_{\mathbf{e}_3}\mathbf{e}_1&=0,\qquad \nabla_{\mathbf{e}_3}\mathbf{e}_2=0,\qquad \nabla_{\mathbf{e}_3}\mathbf{e}_3=0.\end{aligned} \tag{2.7}$$

Moreover we put

$$R_{ijk}=R(\mathbf{e}_i,\mathbf{e}_j)\mathbf{e}_k,\quad R_{ijkl}=R(\mathbf{e}_i,\mathbf{e}_j,\mathbf{e}_k,\mathbf{e}_l),$$

where the indices i,j,k and l take the values $1,2$ and 3.

$$R_{122}=-\mathbf{e}_{1,},\quad R_{133}=-\mathbf{e}_{1,},\ R_{233}=-\mathbf{e}_2,$$

and

$$R_{1212}=\ R_{1313}=R_{2323}=1. \tag{2.8}$$

§3. Biharmonic Curves in the Special Three-Dimensional $\phi-$Ricci Symmetric Para-Sasakian Manifold $\mathbb{P}$

Let us consider biharmonicity of curves in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame field along γ. Then, the Frenet frame satisfies the following Frenet–Serret equations:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= -\kappa\mathbf{T} + \tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N}, \end{aligned} \tag{3.1}$$

where κ is the curvature of γ and τ its torsion.

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, we can write

$$\begin{aligned} \mathbf{T} &= T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3, \\ \mathbf{N} &= N_1\mathbf{e}_1 + N_2\mathbf{e}_2 + N_3\mathbf{e}_3, \\ \mathbf{B} &= \mathbf{T} \times \mathbf{N} = B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{e}_3. \end{aligned} \tag{3.2}$$

Theorem 3.1([12]) *$\gamma : I \longrightarrow \mathbb{P}$ is a biharmonic curve if and only if*

$$\begin{aligned} \kappa &= \text{constant} \neq 0, \\ \kappa^2 + \tau^2 &= 1, \\ \tau' &= 0. \end{aligned} \tag{3.3}$$

Theorem 3.2([12]) *All of biharmonic curves in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$ are helices.*

§4. New Approach for Biharmonic Curves in $\mathbb{P}$

A map

$$\exp : R \times \mathbb{P}_3^3 \to GL\left(3, \mathbb{R}\right) \subset \mathbb{P}_3^3,\ \ (t, \mathcal{A}) \to \exp\left(t, \mathcal{A}\right) = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathcal{A}^k$$

is called exponential map in para-Sasakian Manifold $\mathbb{P}$.

Definition 4.1 $\langle \mathcal{A}, \mathcal{B} \rangle_{\mathbb{P}} = \mathfrak{trace}\left(\mathcal{A}\mathcal{B}^T\right)$ *is called an inner product for $\mathcal{A}, \mathcal{B} \in \mathbb{P}_3^3$.*

Firstly, let us calculate the arbitrary parameter t according to the arclength parameter s. It is well known that

$$s = \int_0^t \left\|\gamma'\left(t\right)\right\|_{\mathbb{P}} dt, \tag{4.1}$$

where

$$\gamma'\left(t\right) = \mathcal{A}\gamma. \tag{4.2}$$

The norm of Equation (4.1), we obtain

$$\|\mathcal{A}\gamma\|_{\mathbb{P}} = \sqrt{-\mathfrak{trace}\,(\mathcal{A}^2)},$$

where $\gamma\gamma^T = I$.

Substituting above equation in (4.1), we have

$$s = \sqrt{-\mathfrak{trace}\,(\mathcal{A}^2)}t$$

Lemma 4.2 *Let $\mathcal{A}$ be a be an anti-symmetric matrix and $n \in \mathbb{N}$. Then,*

i) If n is odd, $\mathcal{A}^n$ is an anti-symmetric matrix.
ii) If n is even, $\mathcal{A}^n$ is a symmetric matrix.
iii) The trace of an anti-symmetric matrix is zero.

The first, second and third derivatives of γ are given as follows:

$$\begin{aligned} \gamma'(s) &= \frac{\mathcal{A}\gamma}{\sqrt{-\mathfrak{trace}\,(\mathcal{A}^2)}}, \\ \gamma''(s) &= \frac{\mathcal{A}^2\gamma}{\left(\sqrt{-\mathfrak{trace}\,(\mathcal{A}^2)}\right)^2}, \\ \gamma'''(s) &= \frac{\mathcal{A}^3\gamma}{\left(\sqrt{-\mathfrak{trace}\,(\mathcal{A}^2)}\right)^3}. \end{aligned} \tag{4.3}$$

§5. Matrix Representation of Biharmonic Curves in Terms of Exponential Maps in the Special Three-Dimensional $\phi-$Ricci Symmetric Para-Sasakian Manifold

Using above sections we obtain following results.

Theorem 5.1 *Let $\gamma : I \longrightarrow \mathbb{P}$ be a unit speed non-geodesic biharmonic curve in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$. Then,*

$$\begin{aligned} \mathcal{A}\gamma &= \sqrt{-\mathfrak{trace}\,(\mathcal{A}^2)}(-\cos\varphi, \sin\varphi e^{-s\cos\varphi + C_1}(\sin[\mathbb{k}s + C] + \cos[\mathbb{k}s + C]), \\ &\quad \sin\varphi e^{-s\cos\varphi + C_1}\sin[\mathbb{k}s + C]), \end{aligned}$$

$$\begin{aligned} \mathcal{A}^2\gamma &= \frac{\left(\sqrt{\mathfrak{trace}\,(\mathcal{A}^4)}\right)}{\kappa}(-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2, \\ &\quad e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2}(\mathbb{k}\sin\varphi\sin[\mathbb{k}s + C] + \cos\varphi\sin\varphi\cos[\mathbb{k}s + C]) \\ &\quad + e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2}(-\mathbb{k}\sin\varphi\cos[\mathbb{k}s + C] + \cos\varphi\sin\varphi\sin[\mathbb{k}s + C]), \\ &\quad - e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2}(-\mathbb{k}\sin\varphi\cos[\mathbb{k}s + C] + \cos\varphi\sin\varphi\sin[\mathbb{k}s + C])), \end{aligned} \tag{5.1}$$

$$\begin{aligned}\mathcal{A}^3\gamma &= \frac{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^3}{\kappa}\left[\frac{\mathfrak{trace}\left(\mathcal{A}^6\right)}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^6}-\frac{\left(\mathfrak{trace}\left(\mathcal{A}^4\right)\right)^2}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^5}\right]^{\frac{1}{2}}(-\sin\varphi e^{-s\cos\varphi+C_1}(\sin[\Bbbk s+C]\\
&+\cos[\Bbbk s+C])e^{-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1 s+\overline{C}_2}.(-\Bbbk\sin\varphi\cos[\Bbbk s+C]+\cos\varphi\sin\varphi\sin[\Bbbk s+C])\\
&-\sin\varphi e^{-s\cos\varphi+C_1}\sin[\Bbbk s+C]\, e^{-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1 s+\overline{C}_2}((\Bbbk\sin\varphi\sin[\Bbbk s+C]+\cos\varphi\sin\varphi\cos[\Bbbk s+C])\\
&+(-\Bbbk\sin\varphi\cos[\Bbbk s+C]+\cos\varphi\sin\varphi\sin[\Bbbk s+C])),\\
&(-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1 s+\overline{C}_2)\sin\varphi e^{-s\cos\varphi+C_1}\sin[\Bbbk s+C]\\
&-\cos\varphi e^{-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1 s+\overline{C}_2}(-\Bbbk\sin\varphi\cos[\Bbbk s+C]+\cos\varphi\sin\varphi\sin[\Bbbk s+C]),\\
&-\cos\varphi e^{-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1 s+\overline{C}_2}((\Bbbk\sin\varphi\sin[\Bbbk s+C]+\cos\varphi\sin\varphi\cos[\Bbbk s+C])\\
&+(-\Bbbk\sin\varphi\cos[\Bbbk s+C]+\cos\varphi\sin\varphi\sin[\Bbbk s+C]))\\
&-\sin\varphi e^{-s\cos\varphi+C_1}(\sin[\Bbbk s+C]+\cos[\Bbbk s+C])(-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1 s+\overline{C}_2)\\
&-\frac{\mathfrak{trace}\left(\mathcal{A}^4\right)}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)}(-\cos\varphi,\sin\varphi e^{x^1}(\sin[\Bbbk s+C]+\cos[\Bbbk s+C]),\sin\varphi e^{x^1}\sin[\Bbbk s+C]).\end{aligned}$$

where C, $\overline{C}_1, \overline{C}_2$ are constants of integration and $\Bbbk=\frac{\sqrt{\kappa^2-\sin^2\varphi}}{\sin\varphi}$.

Proof From (3.1) and Theorem 3.2, imply

$$\mathbf{T}=\sin\varphi\cos[\Bbbk s+C]\,\mathbf{e}_1+\sin\varphi\sin[\Bbbk s+C]\,\mathbf{e}_2+\cos\varphi\mathbf{e}_3,$$

where $\Bbbk=\frac{\sqrt{\kappa^2-\sin^2\varphi}}{\sin\varphi}$.

Using (2.4) in above equation, we obtain

$$\mathbf{T}=(-\cos\varphi,\sin\varphi e^{x^1}(\sin[\Bbbk s+C]+\cos[\Bbbk s+C]),\sin\varphi e^{x^1}\sin[\Bbbk s+C]). \tag{5.2}$$

On the other hand, first equation of (3.3) we have

$$\begin{aligned}\mathcal{A}\gamma &= \sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}(-\cos\varphi,\sin\varphi e^{-s\cos\varphi+C_1}(\sin[\Bbbk s+C]+\cos[\Bbbk s+C]),\\
&\sin\varphi e^{-s\cos\varphi+C_1}\sin[\Bbbk s+C]).\end{aligned}$$

Using Gram-Schmidt method

$$\bar{\mathbf{N}}=\gamma''(s)-\frac{\langle\gamma''(s),\mathbf{T}\rangle_{\mathbb{P}}}{\|\mathbf{T}\|_{\mathbb{P}}^2}\mathbf{T}.$$

Therefore

$$\langle\gamma''(s),\mathbf{T}\rangle_{\mathbb{P}}=<\frac{\mathcal{A}^2\gamma}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^2},\frac{\mathcal{A}\gamma}{\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}}>_{\mathbb{P}}$$

or

$$\langle \gamma''(s), \mathbf{T} \rangle_{\mathbb{P}} = \left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)} \right)^{-3} < \mathcal{A}^2\gamma, \mathcal{A}\gamma > .$$

Also from Definition 3.1 and Lemma 3.2, we obtain

$$< \mathcal{A}^2\gamma, \mathcal{A}\gamma >_{\mathbb{P}} = \mathfrak{trace}\left(-\mathcal{A}^3\right) = 0.$$

Since

$$\bar{\mathbf{N}} = \left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)} \right)^{-2} \mathcal{A}^2\gamma.$$

So we immediately arrive at

$$\mathbf{N} = \frac{\bar{\mathbf{N}}}{\left\|\bar{\mathbf{N}}\right\|_{\mathbb{P}}} = \frac{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)} \right)^{-2} \mathcal{A}^2\gamma}{\left\| \left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)} \right)^{-2} \mathcal{A}^2\gamma \right\|_{\mathbb{P}}} = \frac{\mathcal{A}^2\gamma}{\sqrt{< \mathcal{A}^2\gamma, \mathcal{A}^2\gamma >_{\mathbb{P}}}}. \tag{5.3}$$

Also from Definition 4.1 and Lemma 4.2 we obtain

$$< \mathcal{A}^2\gamma, \mathcal{A}^2\gamma >_{\mathbb{P}} = \mathfrak{trace}\left(\mathcal{A}^4\right). \tag{5.4}$$

Substituting (5.4) in (5.3), we have

$$\mathbf{N} = \left(\sqrt{\mathfrak{trace}\left(\mathcal{A}^4\right)} \right)^{-1} \mathcal{A}^2\gamma.$$

On the other hand, using (2.7), we have

$$\nabla_{\mathbf{T}}\mathbf{T} = \left(T_1' + T_1T_3\right)\mathbf{e}_1 + \left(T_2' + T_2T_3\right)\mathbf{e}_2 + \left(T_3' - \left(T_1^2 - T_2^2\right)\right)\mathbf{e}_3. \tag{5.5}$$

From (4.1) and (5.5), we get

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \sin\varphi\left(-\mathbb{k}\sin\left[\mathbb{k}s + C\right] + \cos\varphi\cos\left[\mathbb{k}s + C\right]\right)\mathbf{e}_1 \\ &\quad + \sin\varphi\left(\mathbb{k}\cos\left[\mathbb{k}s + C\right] + \cos\varphi\sin\left[\mathbb{k}s + C\right]\right)\mathbf{e}_2 \\ &\quad - \sin^2\varphi\mathbf{e}_3, \end{aligned} \tag{5.6}$$

where $\mathbb{k} = \frac{\sqrt{\kappa^2 - \sin^2\varphi}}{\sin\varphi}$. By the use of above equation, we get

$$\begin{aligned} \mathcal{A}^2\gamma &= \left(\sqrt{\mathfrak{trace}\left(\mathcal{A}^4\right)} \right)\mathbf{N} \\ &= \frac{\left(\sqrt{\mathfrak{trace}\left(\mathcal{A}^4\right)} \right)}{\kappa}[(\mathbb{k}\sin\varphi\sin\left[\mathbb{k}s + C\right] + \cos\varphi\sin\varphi\cos\left[\mathbb{k}s + C\right])\mathbf{e}_1 \\ &\quad + (-\mathbb{k}\sin\varphi\cos\left[\mathbb{k}s + C\right] + \cos\varphi\sin\varphi\sin\left[\mathbb{k}s + C\right])\mathbf{e}_2 \\ &\quad - \sin^2\varphi\mathbf{e}_3]. \end{aligned} \tag{5.7}$$

Substituting (2.4) in (5.7), we have

$$\begin{aligned} \mathcal{A}^2\gamma &= \frac{\left(\sqrt{\mathfrak{trace}\left(\mathcal{A}^4\right)} \right)}{\kappa}(-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2, \\ &\quad e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2}\left(\mathbb{k}\sin\varphi\sin\left[\mathbb{k}s + C\right] + \cos\varphi\sin\varphi\cos\left[\mathbb{k}s + C\right]\right) \\ &\quad + e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2}\left(-\mathbb{k}\sin\varphi\cos\left[\mathbb{k}s + C\right] + \cos\varphi\sin\varphi\sin\left[\mathbb{k}s + C\right]\right), \\ &\quad - e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2}\left(-\mathbb{k}\sin\varphi\cos\left[\mathbb{k}s + C\right] + \cos\varphi\sin\varphi\sin\left[\mathbb{k}s + C\right]\right)), \end{aligned} \tag{5.8}$$

where $\overline{C}_1, \overline{C}_2$ are constants of integration.

Using same calculations we get

$$\mathbf{B} = [\frac{\mathfrak{trace}\left(\mathcal{A}^6\right)}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^6} - \frac{\left(\mathfrak{trace}\left(\mathcal{A}^4\right)\right)^2}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^5}]^{-\frac{1}{2}}[\frac{\mathcal{A}^3\gamma}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^3} + \frac{\mathfrak{trace}\left(\mathcal{A}^4\right)}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^5}\mathcal{A}\gamma].$$

From above equation we have

$$\mathcal{A}^3\gamma = \left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^3 [\frac{\mathfrak{trace}\left(\mathcal{A}^6\right)}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^6} - \frac{\left(\mathfrak{trace}\left(\mathcal{A}^4\right)\right)^2}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^5}]^{\frac{1}{2}}\mathbf{B} - \frac{\mathfrak{trace}\left(\mathcal{A}^4\right)}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^2}\mathcal{A}\gamma.$$

Cross product of $\mathbf{T} \times \mathbf{N} = \mathbf{B}$ gives us

$$\begin{aligned}
\mathcal{A}^3\gamma \quad = \quad & \frac{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^3}{\kappa}[\frac{\mathfrak{trace}\left(\mathcal{A}^6\right)}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^6} - \frac{\left(\mathfrak{trace}\left(\mathcal{A}^4\right)\right)^2}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^5}]^{\frac{1}{2}}(-\sin\varphi e^{-s\cos\varphi + C_1}(\sin[\Bbbk s + C] \\
& + \cos[\Bbbk s + C])e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2}.(-\Bbbk\sin\varphi\cos[\Bbbk s + C] + \cos\varphi\sin\varphi\sin[\Bbbk s + C]) \\
& - \sin\varphi e^{-s\cos\varphi + C_1}\sin[\Bbbk s + C]\, e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2}((\Bbbk\sin\varphi\sin[\Bbbk s + C] + \cos\varphi\sin\varphi\cos[\Bbbk s + C]) \\
& + (-\Bbbk\sin\varphi\cos[\Bbbk s + C] + \cos\varphi\sin\varphi\sin[\Bbbk s + C])), \\
& (-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2)\sin\varphi e^{-s\cos\varphi + C_1}\sin[\Bbbk s + C] \qquad (5.9) \\
& - \cos\varphi e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2}\left(-\Bbbk\sin\varphi\cos[\Bbbk s + C] + \cos\varphi\sin\varphi\sin[\Bbbk s + C]\right), \\
& - \cos\varphi e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2}((\Bbbk\sin\varphi\sin[\Bbbk s + C] + \cos\varphi\sin\varphi\cos[\Bbbk s + C]) \\
& + (-\Bbbk\sin\varphi\cos[\Bbbk s + C] + \cos\varphi\sin\varphi\sin[\Bbbk s + C])) \\
& - \sin\varphi e^{-s\cos\varphi + C_1}\left(\sin[\Bbbk s + C] + \cos[\Bbbk s + C]\right)(-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2) \\
& - \frac{\mathfrak{trace}\left(\mathcal{A}^4\right)}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)}(-\cos\varphi, \sin\varphi e^{x^1}\left(\sin[\Bbbk s + C] + \cos[\Bbbk s + C]\right), \sin\varphi e^{x^1}\sin[\Bbbk s + C]).
\end{aligned}$$

So the proof is completed. □

In the light of Theorem 5.1, we express the following corollary without proof.

Corollary 5.2

$$\mathcal{A}^3\gamma = [\frac{\mathfrak{trace}\left(\mathcal{A}^6\right)}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^3} - \frac{\left(\mathfrak{trace}\left(\mathcal{A}^4\right)\right)^2}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^2}]^{\frac{1}{2}}\mathbf{B} - \frac{\mathfrak{trace}\left(\mathcal{A}^4\right)}{\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}}\mathbf{T}.$$

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On Square Difference Graphs

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Abstract: In graph theory number labeling problems play vital role. Let $G = (V, E)$ be a (p, q)-graph with vertex set V and edge set E. Let f be a vertex valued bijective function from $V(G) \to \{0, 1, ..., p-1\}$. An edge valued function f^* can be defined on G as a function of squares of vertex values. Graphs which satisfy the injectivity of this type of edge valued functions are called square graphs. Square graphs have two major divisions: they are square sum graphs and square difference graphs. In this paper we concentrate on square difference or SD graphs. An edge labeling f^* on $E(G)$ can be defined as follows. $f^*(uv) = \left|(f(u))^2 - (f(v))^2\right|$ for every uv in $E(G)$. If f^* is injective, then the labeling is said to be a SD labeling. A graph which satisfies SD labeling is known as a SD graph. We illuminate some of the results on number theory into the structure of SD graphs. Also, established some classes of SD graphs and established that every graph can be embedded into a SD graph.

Key Words: SD labeling, SD graph, strongly SD graph, perfect SD graph.

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§1. Introduction

The research in graph enumeration and graph labeling started way back in 1857 by Arthur Cayley. Graph labeling and enumeration finds the application in chemical graph theory, social networking and computer networking and channel assignment problem. Abundant literature exists as of today concerning the structure of graphs admitting a variety of functions assigning real numbers to their elements so that certain given conditions are satisfied. Here we are

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interested the study of vertex functions $f : V(G) \to \{0, 1, \cdots, p-1\}$ for which an edge valued injective function f^* can be defined on G as function of squares of vertex values. Graphs which satisfy this type of labeling are called square graphs. Square graphs have two major divisions: they are square sum graphs and square difference graphs.

In this paper, we concentrate on square difference or SD graphs. This new type of labeling of graphs is closely related to the equation $x^2 - y^2 = n$. It is important to note that certain numbers like $6, 10, 14$ etc., cannot be written as the difference of two squares. Hence it is very interesting to study those graphs which takes the first consecutive numbers that can be expressed as the difference of two squares. Here, consider only a finite undirected graph without loops or multiple edges. Terms not specifically defined in this paper may be found in Harrary (1969), [6] and all the number theoretic results used here found in [3,4] and [2,8].

Here we recall some results of number theory, which are essential for our study.

Definition 1.1 *An integer is said to be representable if it can be represent as difference of two squares.*

Theorem 1.2([7]) *The product of any number of representable integers is also representable.*

Theorem 1.3([7]) *Every square integer is of the form following*

(*i*) $4q$ *or* $4q + 1$;

(*ii*) $5q,\ 5q + 1$ *or* $5q - 1$.

Theorem 1.4([7]) *If* $n = x^2 - y^2$, *then* $n \equiv 0, 1, 3 (\mathrm{mod} 4)$.

Corollary 1.5([7]) *An odd number is a difference of two successive squares.*

Theorem 1.6([7]) *The difference of squares of consecutive numbers is equal to the sum of the numbers.*

§2. SD Graphs

Definition 2.1 *Let* $G = (V, E)$ *be a* (p, q)*-graph with vertex set* V *and edge set* E. *Let* f *be a vertex valued bijective function from* $V(G) \to \{0, 1, \cdots, p-1\}$. *An edge valued function* f^* *can be defined on* G *as* $f^*(uv) = \left|(f(u))^2 - (f(v))^2\right|$ *for every* uv *in* $E(G)$. *If* f^* *is injective, then the labeling is said to be a* SD *labeling. A graph which satisfies* SD *labeling is known as an* SD *graph.*

Example 2.2 An example of SD graphs is given below.

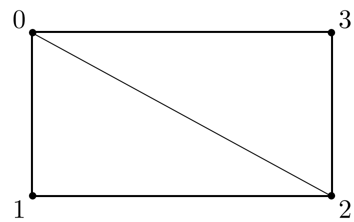

Fig.1

The following are some simple observations obtained immediately from the definition of SD graph.

Observation 2.3 *For every edge $e = uv \in E(G)$ then $f^*(e) \equiv 0, 1, 3(\text{mod}4)$.*

Observation 2.4 *If $e = uv \in E(G)$ with $f(u) = 0$ then $f^*(e) \equiv 0, 1(\text{mod}4)$.*

Observation 2.5 *For a SD graph the product of edge values is a representable number.*

Observation 2.6 *Let G be a (p, q)-graph with a square difference labeling f then $[f(u)]^2$ occurs $d(u)$ times in the sum $\sum f^*(e)$. Take the p vertices of G as $u_1, u_2, ...u_p$ and assign values to them in such a way that $f(u_1) < f(u_2) < ... < f(u_p)$, then for all $i = 0, 1, \cdots, p-1$ we have $\sum f^*(e) = d(u_p)[f(u_p)]^2 + \sum\limits_{i=1}^{p-1} (k_i - l_i)[f(u_i)]^2$, where for each edge u_iu_j,*

$k_i =$ number of vertices $u_j$with $f(u_i) > f(u_j), i \neq j$

$l_i =$ number of vertices $u_j$with $f(u_i) < f(u_j), i \neq j$

In particular if $k_i = l_i$ for all i, we get the above result as $\sum f^*(e) = d(u_p)[f(u_p)]^2$.

Theorem 2.7 *Let G be a connected SD graph with an SD labeling f. Then $f^*(e) \equiv 1(mod\ 2)$ for at least one edge $e \in E(G)$. Further if $f^*(e) \equiv 1(mod\ 2)$, $\forall e \in E(G)$, then G is bipartite.*

Proof Let $X = \{u : f(u) \text{ is even }\}$ and $Y = \{v : f(v) \text{ is odd }\}$. Since G is connected there exists at least one edge $e = uv$ such that $u \in X$ and $v \in Y$. Hence $f^*(e) \equiv 1(mod\ 2)$. If $f^*(e) \equiv 1(mod\ 2)$, $\forall e \in E(G)$, it follows that $f(u)$ and $f(v)$ are of opposite parity and X and Y form a bipartition of G and G is a bipartite graph. □

§3. Some classes of SD graphs

Theorem 3.1 *The graph $G = K_2 + mK_1$ is an SD graph.*

Proof Let $V(G) = \{u_1, u_2, ...u_{m+2}\}$ where $V(K_2) = \{u_1, u_2\}$. Define $f : V(G) \to \{0, 1, ..., m+1\}$ by $f(u_i) = i - 1, 1 \leqslant i \leqslant m+2$. Clearly, the induced function f^* is injective, for if $f^*(u_1u_i) = f^*(u_2u_j)$ then, $|[f(u_1)]^2 - [f(u_i)]^2| = [|f(u_2)]^2 - [f(u_j)]^2|$. Since $f(u_1) = 0$ and $f(u_2) = 1$, we get $(f(u_i))^2 = (f(u_j))^2 - 1$ so that either $f(u_i) = 0$ or $f(u_j) = 0$, which is a contradiction. Hence f^* is injective and G is an SD graph. □

Theorem 3.2 *Every star is an SD graph.*

Proof Let $V(K_{1,n}) = \{u_1, u_2, \cdots, u_n, u_{n+1}\}$ where $e_i = u_1u_i$ for $2 \leqslant i \leqslant n$. Define $f : V(K_{1n}) \to \{0, 1, \cdots, n-1\}$ as $f(u_1) = 0$ and $f(u_i) = i - 1$. Then $f^*(E(K_{1,n})) = \{1^2, 2^2, \cdots, (n-1)^2\}$ and hence f^* is infective and f is a SD labeling on K_{1n}. Hence every star is an SD graph. □

Theorem 3.3 *Every path is an SD graph.*

Proof Let $P_n = (u_1, u_2, \cdots, u_n)$ where $e_i = u_iu_{i+1}$ for $1 \leqslant i \leqslant n$. Define $f : V(G) \to \{0, 1, \cdots, n-1\}$ as $f(u_i) = i - 1$. Then by the Theorem **??** $f^*(E(P_n)) = \{1, 3, \cdots, 2n-3\}$ and hence f^* is infective and f is a SD labeling on P_n. Hence every path is an SD graph. □

Theorem 3.4 *A complete graph K_n is SD if and only if $n \leqslant 5$.*

Proof The SD labeling of the complete graph K_n for $n \leqslant 5$ is given in figures 2 and 5. Further since $5^2 - 4^2 = 3^2 - 0^2$ it follows that $K_n,\ n \geqslant 6$ is not an SD graph. □

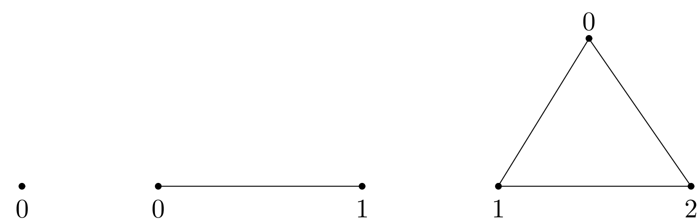

Fig.2

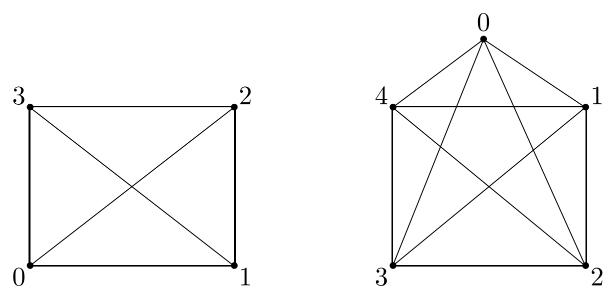

Fig.3

Theorem 3.5 *Every cycle is an SD graph.*

Proof Let $C_n = (u_1, u_2, \cdots, u_n)$ where $e_i = u_iu_{i+1}$ for $1 \leqslant i \leqslant n-1$ and $e_n = u_nu_1$. Define $f : V(G) \to \{0, 1, \cdots, n-1\}$ as $f(u_i) = i - 1$ for $1 \leqslant i \leqslant n$. Also $f^*(E(C_n)) = \{1, 3, \cdots, 2n-3, (n-1)^2\}$. If n is odd then $(n-1)^2$ is even, so f^* is infective and f is a SD labeling on C_n. If n is even, since $(n-1)^2 > 2n-3$ for $n \geqslant 3$, then also f^* is injective and f is a SD labeling on C_n. Hence cycles are SD graphs. □

Theorem 3.6 *Every friendship graph $C_3^{(n)}$ is an SD graph.*

Proof Let $G_1, G_2, \cdots G_n$ be the n copies of C_3, all concatenated at exactly one vertex say, z. Let $G_i = (z, u_{i1}, u_{i2})\,, 1 \leqslant i \leqslant n$. Define $f : V(C_3^{(n)}) \to \{0, 1, \cdots, 2n\}$ as $f(z) = 0, f(u_{i1}) = i$ and $f(u_{i2}) = n + i$.
Then $f^*\left(E(C_3^{(n)})\right) = \left\{1^2, 2^2, \cdots, (2n)^2, n^2 + 2n, n^2 + 4n, n^2 + 6n, \cdots, 3n^2\right\}$ and the induced edge function f^* is injective and f is an SD labeling on $C_3^{(n)}$. Hence the proof. □

Definition 3.7([1]) *A cycle-cactus is such a graph that consisting of n copies of C_k, $k \geq 3$ concatenated at exactly one vertex is denoted as $C_k^{(n)}$.*

Theorem 3.8 *A complete bipartite graphs $K_{m,n}$ is SD if $m \leq 4$ for any integer $n \geq 1$.*

Proof Let $X = \{x_1, x_2, \cdots, x_m\}$,$Y = \{y_1, y_2, \cdots, y_n\}$ be the partition of $K_{m,n}$. Define a vertex labeling as follows:

(1) If $m = 1$, then $f(x_1) = 0$ and $f(y_i) = i, 1 \leqslant i \leqslant n$;

(2) If $m = 2$, then $f(x_1) = 0, f(x_2) = 1$ and $f(y_i) = i + 1, 1 \leqslant i \leqslant n$;

(3) If $m = 3$, then $f(x_1) = 0, f(x_2) = 1, f(x_3) = 2$ and $f(y_i) = i + 2, 1 \leqslant i \leqslant n$;

(4) If $m = 4$, then $f(x_1) = 0, f(x_2) = 1, f(x_3) = 3, f(x_4) = 5$ and $f(y_i) = 2i$, if $i = 1, 2, 3$, $f(y_i) = 2i - 1$, if $i = 4$, and $f(y_i) = i + 3$, if $i > 4$.

It is easy to verify that the edge valued function f^* is injective on $E(K_{m,n})$ and hence the theorem. □

Theorem 3.9 *Every triangle snake is an SD graph.*

Proof Let G be a triangle snake with $2n + 1$ vertices and let the vertex set $V(G)$ be $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_{n+1}\}$ and the edge set $E(G)$ be $\{u_iv_i, u_iv_{i+1}, v_iv_{i+1}\}$, where $1 \leqslant i \leqslant n$. We can define a vertex labeling $f : V(G) \to \{0, 1, 2, \cdots, 2n\}$ as $f(u_i) = 2i - 1, 1 \leqslant i \leqslant n$ and $f(v_i) = 2(i - 1), 1 \leqslant i \leqslant n + 1$. Then the corresponding edge function f^* is injective and hence all triangle snakes are SD graphs. □

Example 3.10 The following example gives an illustration for the above theorem.

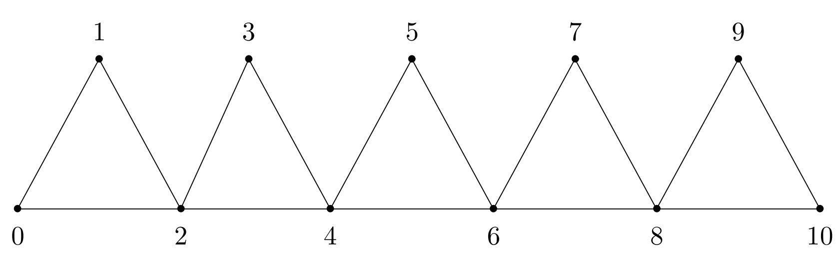

Fig.4

Observation 3.11 P_4 is a connected SD graph with prime edge labels.

Proof A prime number is a representable number if and only if it can be represent as difference of two consecutive integers. Hence there exist a SD labelling on P_4 with prime edge values as follows

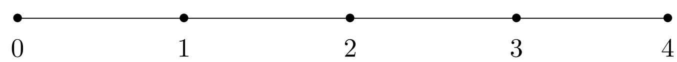

Fig.5

This completes the proof. □

Corollary 3.12 *A connected* 5 *vertexed graph with prime edge values is isomorphic to* P_4.

Corollary 3.13 *There is not exist a connected* (p, q) - SD *graph with prime edge values if* $p > 5$.

Conjecture 3.14 *Every tree is an* SD *graph.*

Conjecture 3.15 *Every cycle-cactus* $C_k^{(n)}$ *is an* SD *graph.*

Conjecture 3.16 *Every complete bipartite graph is an* SD *graph.*

§4. Strongly SD Graphs

Definition 4.1 *An* SD *graph is said to be a* strongly SD *if the edge values are consecutive representable numbers.*

In other words, an SD graph $G = (V, E)$ is said to be a strongly SD if $f^*(E(G))$ consists the first q consecutive numbers of the form $a^2 - b^2, a \leqslant p-1, b \leqslant p-1, a \neq b$ and the corresponding labeling is said to be a strongly SD labeling of G.

Example 4.2 An example of a strongly SD graph is given below.

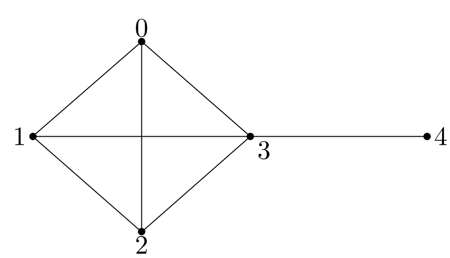

Fig.6

Theorem 4.3 *Every strongly* SD *graph except* K_1, K_2 *and* $K_{1,2}$ *contain at least one triangle.*

Proof Clearly K_1, K_2 and $K_{1,2}$ are triangle free strongly SD graphs. For a strongly square difference graph with three edges, the possible edge values are $1, 3$ and 4. To obtain the edge value 3, the vertices which label 1 and 2 should be adjacent. Similarly to obtain the edge value 4, the vertices which label 0 and 2 should be adjacent. In this case the graph should contains a triangle $(uvwu)$ with $f(u) = 0, f(v) = 1$ and $f(w) = 2$. □

Corollary 4.4 *All cycles $C_n, n \geqslant 4$ are not strongly SD graphs.*

Corollary 4.5 *A complete bipartite graph $K_{m,n}$ is a strongly SD graph if and only if $m = 1,\ n \leqslant 2$.*

Theorem 4.6 *A unicyclic graph is strongly SD if and only if it is either C_3 or C_3 with one pendant edge or C_3 with a path of length 2.*

Proof Clearly, C_3, C_3 with one pendant edge and are strongly SD graphs. Suppose G is a unicyclic graph which is strongly SD. Then $f^*(E(G)) = \{1, 3, 4, 5, 7, 8, \cdots\}$.

If $f^*(E(G)) = \{1, 3, 4\}$, then $G \cong C_3$;

If $f^*(E(G)) = \{1, 3, 4, 5\}$, then G is isomorphic to C_3 along with one pendant edge;

If $f^*(E(G)) = \{1, 3, 4, 5, 7\}$, then G is isomorphic to C_3 along with a path of length 2;

If $f^*(E(G)) = \{1, 3, 4, 5, 7, 8\}$, then to obtain the edge value 8, the vertices with labels 3 and 1 should be adjacent. In this case one more triangle will be formed, a contradiction since G is unicyclic.

Therefore the unicyclic graphs which admits a strongly SD labeling are either C_3 or C_3 with one pendant edge or C_3 with a path of length 2. □

Conjecture 4.7 *Every cycle $C_n, n \geqslant 4$ can be embedded as an induced subgraph of a strongly SD graph.*

Problem 4.8 *Find the number of strongly SD graphs for a given number of edges.*

Problem 4.9 *Characterize the strongly SD graphs.*

By the definition of SD graphs it is clear that a big family of graphs are not SD. Hence on SD graphs embedding theorems have an important role to play. In the following section we proved an embedding theorem.

§5. Embeddings on SD Graphs

Theorem 5.1 *Every $(p, q)-$ graph G can be embedded into a connected SD graph.*

Proof Let G be a graph with vertex set $V(G) = \{u_1, u_2, \cdots, u_p\}$. We shall establish an embedding of G in H, where H is a graph with $|V(H)| = 5^{p-1} + 1$ and $|E(H)| = 5^{p-1} + q - p$. Label the vertices of G by $f(u_1) = 0, f(u_2) = 1$ and $f(u_i) = 5^{i-1}$, $3 \leqslant i \leqslant p$. Let $v_1, v_2, \cdots, v_n$ be the isolated vertices where $n = 5^{p-1} + 1 - p$. Let $X = \left\{1, 2, \cdots, 5^{p-1} - 1\right\}$. Label the

vertices v_i with the numbers from the set $X - \{5, 5^2, \cdots, 5^{p-1}\}$. The graph H obtained from G as follows. Join $u_1$to v_k if $f(v_k)$ is odd for $1 \leqslant k \leqslant n$ and $u_2$to v_l if $f(v_l)$ is even for $1 \leqslant l \leqslant n$. Clearly f is a bijection from $V(H) \to \{0, 1, \cdots, 5^{p-1}\}$. Now we prove that the induced edge labeling f^* is injective. Assume $f^*(e_1) = f^*(e_2)$ for some $e_{1,}e_2 \in E(G)$. Since f is injective it follows that e_1 and e_2 are non adjacent. Suppose $e_1 = u_k u_l$ and $e_2 = u_i u_j$ where $1 \leqslant i \leqslant j, k \leqslant l \leqslant p$. Then $(5^j)^2 - (5^l)^2 = (5^l)^2 + (5^k)^2$ from which we obtain $(5^i)^2 + (5^k)^2 = (5^l)^2 + (5^j)^2$. If $u_i = u_1$ then $(5^k)^2 = (5^l)^2 + (5^j)^2$ which is a contradiction. Otherwise dividing both sides of the equation by $(5a)^2$, where $a = \min(i, j, k, l)$, we get an edge where one side is congruent to 1(mod)5 and the other side is congruent to 0(mod)5, which is a contradiction. If $e_1 = u_k u_l$ and $e_2 = u_1 v_i$ then $f^*(e_1) = f^*(e_2) \Rightarrow \left(5^l\right)^2 - \left(5^k\right)^2 = (f(v_i))^2$ if $f(v_i)$ is odd and $\left(5^l\right)^2 - \left(5^k\right)^2 = (f(v_i))^2 - 1$ if $f(v_i)$ is even. In both case we get contradictions. Hence the induced function f^* is a SD labeling and the graph H is a SD graph which contains G as a subgraph. □

§6. Perfect SD Graphs

Definition 6.1 *An SD graph is said to be a perfect SD graph if the edge values are consecutive perfect squares. The corresponding SD labelling is said to be a perfect SD labelling.*

Example 6.2

Fig.7

Observation 6.3 *Every star is a perfect SD graph.*

Observation 6.4 *The cycles C_3, C_4 and C_5 are not perfect SD graphs.*

Observation 6.5 *Every friendship graph $C_3^{(n)}$ is not a perfect SD graph.*

Conjecture 6.6 *Every cycle is not a perfect SD graph.*

Theorem 6.7 *These graphs C_3, C_4 and C_5 can be embedded into perfect SD graph.*

Proof The following embeddings give the proof. □

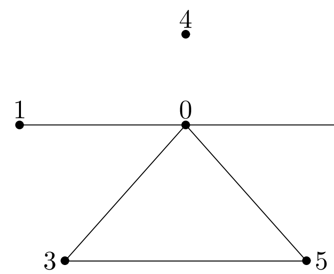

Fig.8 **The embedding of** C_3

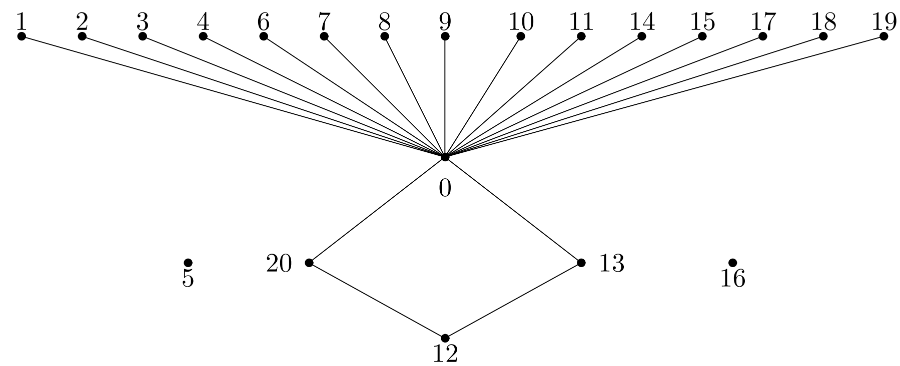

Fig.9 **The embedding of** C_4

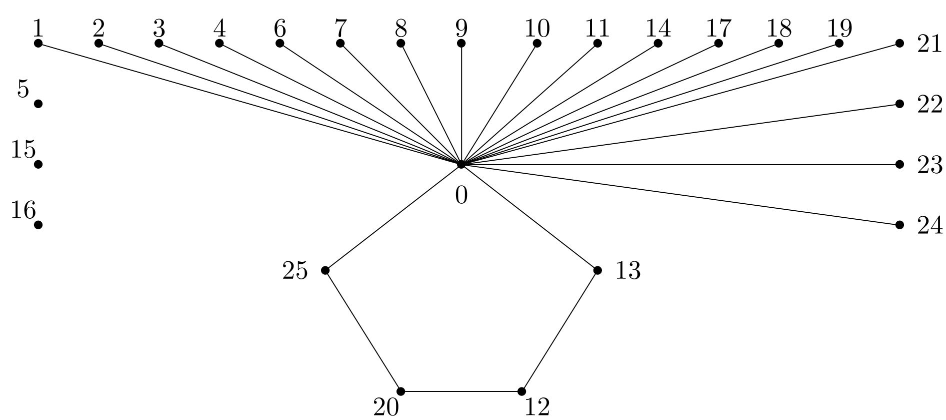

Fig.10 **The embedding of** C_5

Conjecture 6.8 *Every cycle can be embedded into a perfect SD graph.*

Theorem 6.9 *The friendship graph* $C_3^{(n)}$ *can be embedded into a perfect SD graph.*

Proof Let $G_1, G_2, \cdots G_n$ be the n copies of C_3, all concatenated at exactly one vertex say, z. Let $G_i = (z, u_{i1}, u_{i2}), 1 \leqslant i \leqslant n$. Define $f : V(C_3^{(n)}) \to \{0, 1, \cdots, 2n\}$ as $f(z) = 0$, $f(u_{i1}) = 2i^2+2i+1$ and $f(u_{i2}) = 2i+1$. Introduce $2n^2+1$ new vertices V_k where $1 \leqslant k \leqslant 2n^2+1$. Join the vertex z to V_k where $k \neq 2i(i+1)$, $1 \leqslant i \leqslant n$.
Then f is a perfect SD labeling on the embedded graph H and hence H is a perfect SD graph. □

Problem 6.10 *Every cycle-cactus $C_k^{(n)}$ can be embedded into a perfect SD graph.*

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On Finsler Spaces with Unified Main Scalar (LC) of the Form $L^2C^2 = f(y) + g(x)$

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Abstract: In the year 1979, M. Matsumoto was studying Finsler spaces with vanishing T-tensor and come to know that for such Finsler spaces L^2C^2 is a function of x only. Later on in the year 1980, Matsumoto with Numata concluded that the condition $L^2C^2 = f(x)$ is not sufficient for vanishing of T-tensor. F. Ikeda in the year 1984, studied Finsler spaces whose L^2C^2 is function of x in detail. In the present paper we shall discuss a Finsler space for which L^2C^2 is a function of x and y ($y^i = \dot{x}^i$) in the form $L^2C^2 = f(y) + g(x)$.

Key Words: Unified main scalar, Berwald space, Landsberg space, T-tensor, C-Reducible Finsler space, Two-dimensional Finsler space.

AMS(2010): 53B40, 53C60

§1. Introduction

In the paper [7] it has been shown that if a Finsler space satisfies the T-condition, then the unified Main Scalar L^2C^2 of the Finsler space M^n is reduced to the function of the position only (i.e. $L^2C^2 = f(x)$), where L is the fundamental function and C^2 is the square of length of the torsion vector C_i. F. Ikeda in the paper [1] has worked out, whether the condition $L^2C^2 = f(x)$ is equivalent to the T-condition or not and considered the properties of such Finsler spaces in detail.

In the present paper, we shall study the T-tensor of such a Finsler space with the condition $L^2C^2 = f(y) + g(x)$. As F. Ikeda in the paper [1], we have also obtained the condition for such a Finsler space ($L^2C^2 = f(y) + g(x)$) to be a Landsberg or Berwald space so that a Landsberg space (resp. Berwald space) satisfying the condition $L^2C^2 = f(y) + g(x)$ reduces to a Berwald space.

The terminology and notation are referred to the Matsumoto's monograph [5].

§2. The Condition $L^2C^2 = f(y) + g(x)$

Let l_i, h_{ij} and C_{ijk} denote the unit vector (i.e. $l_i = \dfrac{y_i}{L}$), the angular metric tensor and the (h)hv-torsion tensor (the Cartan torsion tensor), respectively. The T-tensor T_{ijkl} is defined by

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[5] [pp. 188, equ. (28.20)] $T_{ijkl} = LC_{ijk}|_l + C_{ijk}l_l + C_{ijl}l_k + C_{ilk}l_j + C_{ljk}l_i$ and the torsion vector C_i is given by $C_i = g^{jk}C_{ijk}$, where the symbol $|_l$ denotes the v-covariant differentiation and g^{jk} is the reciprocal tensor of g_{jk}.

Assume that the function L^2C^2 is a non-zero function of position and direction s.t. $L^2C^2 = f(y) + g(x)$. Differentiation of this equation by y^i yields

$$L^2C^2|_i + 2C^2y_i = f_i \tag{1}$$

where the symbol $|_i$ denotes the differentiation by $y^i and f_i = \dfrac{\partial f}{\partial y^i}$. Since, $C^2 = g^{ij}C_iC_j$ and $T_{ij}(= g^{kl}T_{ijkl}) = LC_i|_j + C_il_j + C_jl_i$. Since, $C^2 = g^{ij}C_iC_j$, then

$$C^2|_h = 2g^{ij}C_i|_hC_j = 2C^iC_i|_h \tag{2}$$

From (1) and (2), we have

$$2C^2Ll_i + 2L^2C^hC_h|_i = f_i \tag{3}$$

Since $T_{ij} = LC_i|_j + l_iC_j + l_jC_i$

$$C^iT_{ih} = LC^iC_i|_h + C^il_iC_h + C^il_hC_i$$

$$C^iT_{ih} = LC^iC_i|_h + C^2l_h \tag{4}$$

From (4) and (3), we get,

$$2LC^iT_{ih} = f_h \tag{5}$$

Conversely, let $2LC^iT_{ih} = f_h$

$$2LC^i(LC_i|_h + l_iC_h + l_hC_i) = f_h$$

$$2LC^iC_i|_hC^i + 2LC^2l_h = f_h$$

$$(L^2C^2)|_h = f_h$$

Integrating, we get

$$L^2C^2 = f(y) + g(x) \tag{6}$$

where, g(x) is a arbitrary function of x only. Thus, we have,

Theorem 2.1 *If for a n-dimensional Finsler space unified scalar L^2C^2 is of the form $L^2C^2 = f(y) + g(x)$, if and only if the T-tensor satisfies the condition $T_{ij}C^j = \dfrac{f_i}{2L}$.*

Again, for a two-dimensional Finsler space the T-tensor [4,5,7,8] can be written as

$$T_{hijk} = I_{;2}m_hm_im_jm_k \text{ and } LC_{ijk} = Im_im_jm_k \text{ this implies } LC = I$$

Since, $(L^2C^2)|_i = f_i$, this implies

$$2LC(LC)|_i = f_i$$

Thus, we have

$$T_{hijk} = \frac{f_r m^r}{2LC} m_h m_i m_j m_k$$

Corollary 2.1 *In a two-dimensional Finsler space with the unified scalar $L^2C^2(L^2C^2 = f(y) + g(x))$ satisfies T-condition if and only if f_i is parallel to l_i i.e. $f_i = \lambda l_i$ for some scalar function λ.*

Now, $f_i = \lambda l_i$ Differentiating above equation with respect to y^j, we get

$$\frac{\partial f_i}{\partial y^j} = \frac{\partial \lambda}{\partial y^j} l_i + \lambda L^{-1}(h_{ij})$$

contracting above equation with respect to y^i, we get

$$\frac{\partial f_i}{\partial y^j} y^i = L \frac{\partial \lambda}{\partial y^j}$$

this implies

$$L \frac{\partial \lambda}{\partial y^j} = -f_j \qquad [\text{Since } f_i y^i = \frac{\partial f}{\partial y^i} y^i = 0 \Longrightarrow \frac{\partial (f_i y^i)}{\partial y^j} = \frac{\partial f_i}{\partial y^j} y^i + f_i \delta^i_j]$$

this implies

$$L \frac{\partial \lambda}{\partial y^j} = -f_j = -\lambda l_j$$

Integrating we get

$$\lambda = \frac{h(x)}{L}$$

where, $h(x)$ is any arbitrary function of x. Again,

$$f_i = \frac{h(x)}{L} l_i = h(x) \frac{\partial L}{L \partial y^i} \text{ or}$$

$$\frac{\partial f}{\partial y^i} = \frac{h(x)}{h(x)L} \frac{\partial (Lh(x))}{\partial y^i}$$

Integrating above equation, we get

$$f(y) = h(x) log(Lh(x)) + p(x) \tag{7}$$

where, p(x) is also any arbitrary function of x. From (7), we have

$$L = \frac{1}{h(x)} e^{\frac{f(y)-p(x)}{h(x)}}$$

Thus, we have

Theorem 2.2 *If a two-dimensional Finsler space with $L^2C^2 = f(y) + g(x)$ satisfies T-condition then the metric function L is given by $L = \frac{1}{h} e^{\frac{f-p}{h}}$, where scalars h and p are arbitrary function of x only.*

In C-reducible Finsler space the T-tensor [5] can be written as,

$$T_{hijk} = \frac{LC^*}{n^2-1}\pi_{hijk}(h_{hi}h_{jk}) \tag{8}$$

where, $C^* = g^{ij}C_i|_j$ and π_{hijk} represents cyclic permutation of the indices h,i,j,k. contracting (8) by g^{jk}, we get

$$T_{hi} = \frac{LC^*}{n-1}h_{hi}$$

this implies

$$f_h = \frac{2L^2C^*}{n-1}C_h$$

Thus,

$$f_hC^h = \frac{2L^2C^2C^*}{n-1} \tag{9}$$

Corollary 2.2 *For a n-dimensional C-reducible Finsler space with unified scalar $L^2C^2 = f(y)+g(x)$ satisfies T-condition if f_i is perpendicular to C^i.*

§3. Landsberg and Berwald Spaces Satisfying Condition $L^2C^2 = f(y)+g(x)$

Hereafter, assume that a Finsler space M^n satisfies the condition $L^2C^2 = f(y)+g(x)$. From the equation (1), the important tensors which will be used later are given by,

$$g_{ij} = \frac{f_{ij} - L^2C^2|_i|_j}{2C^2} - \frac{1}{LC^2}(f_il_j + f_jl_i) + yl_il_j \tag{10}$$

$$\begin{aligned} C_{ijk} \quad = \quad & \frac{f_{ijk} - L^2C^2|_i|_j|_k}{4C^2} + \frac{2}{L}(h_{ij}l_k + h_{jk}l_i + h_{ki}l_j - 3l_il_jl_k) - \\ & \frac{1}{2LC^2}(f_{ik}l_j + f_{jk}l_i + f_{ij}l_k) + \frac{1}{2L^2C^2}\pi_{(ijk)}(f_i(h_{jk} - f_{jk})) - \\ & \frac{1}{2L^2C^2}(f_il_jl_k + f_jl_il_k + f_kl_il_j) - \frac{1}{L^3C^2}(f_if_jl_k + f_jf_kl_i + f_kf_il_j) \end{aligned} \tag{11}$$

$$\begin{aligned} C_{ijk|h} \quad = \quad & \frac{f_{ijk|h} - L^2C^2|_i|_j|_{k|h}}{4C^2} - \frac{C^2_{|h}}{C^2}C_{ijk} - \frac{1}{2LC^2}(f_{ik|h}l_j + \\ & f_{jk|h}l_i + f_{ij|h}l_k) + \frac{1}{2L^2C^2}\pi_{(ijk)}(f_{i|h}h_{jk}) - \frac{1}{2L^2C^2}(f_{ij|h}f_k + \\ & f_{ij}f_{k|h} + f_{jk|h}f_i + f_{jk}f_{i|h} + f_{ik|h}f_j + f_{ik}f_{j|h}) - \frac{1}{2L^2C^2}(f_{i|h}l_jl_k + \\ & f_{j|h}l_il_k + f_{k|h}l_il_j) - \frac{1}{L^3C^2}(f_{i|h}f_jl_k + f_if_{j|h}l_k + f_{j|h}f_kl_i + \\ & f_jf_{k|h}l_i + f_{k|h}f_il_j + f_kf_{i|h}l_j) \end{aligned} \tag{12}$$

Contracting above equation by y^h, we get

$$\begin{aligned} P_{ijk} \;=\; & \frac{f_{ijk|0} - L^2C^2|_i|_j|_{k|0}}{4C^2} - \frac{C^2_{|0}}{C^2}C_{ijk} - \frac{1}{2LC^2}(f_{ik|0}l_j + \\ & f_{jk|0}l_i + f_{ij|0}l_k) + \frac{1}{2L^2C^2}\pi_{(ijk)}(f_{i|0}h_{jk}) - \frac{1}{2L^2C^2}(f_{ij|0}f_k + \\ & f_{ij}f_{k|0} + f_{jk|0}f_i + f_{jk}f_{i|0} + f_{ik|0}f_j + f_{ik}f_{j|0}) - \frac{1}{2L^2C^2}(f_{i|0}l_jl_k + \\ & f_{j|0}l_il_k + f_{k|0}l_il_j) - \frac{1}{L^3C^2}(f_{i|0}f_jl_k + f_if_{j|0}l_k + f_{j|0}f_kl_i + \\ & f_jf_{k|0}l_i + f_{k|0}f_il_j + f_kf_{i|0}l_j) \end{aligned} \tag{13}$$

where, P_{ijk} is the (v)hv-torsion tensor, the symbol $_{|i}$ denotes the h-covariant differentiation and the index '0' means the contraction by y^i.

The above equation (12) (resp. 13) gives the result that the condition $C_{ijk|l} = 0$ $(res.P_{ijk} = 0)$ is equivalent to equation (14) (resp.15).

$$\begin{aligned} & \frac{f_{ijk|h} - L^2C^2|_i|_j|_{k|h}}{4C^2} - \frac{C^2_{|h}}{C^2}C_{ijk} - \frac{1}{2LC^2}(f_{ik|h}l_j + f_{jk|h}l_i + f_{ij|h}l_k) \\ & + \frac{1}{2L^2C^2}\pi_{(ijk)}(f_{i|h}h_{jk}) - \frac{1}{2L^2C^2}(f_{ij|h}f_k + f_{ij}f_{k|h} + f_{jk|h}f_i + f_{jk}f_{i|h} \\ & + f_{ik|h}f_j + f_{ik}f_{j|h}) - \frac{1}{2L^2C^2}(f_{i|h}l_jl_k + f_{j|h}l_il_k + f_{k|h}l_il_j) \\ & - \frac{1}{L^3C^2}(f_{i|h}f_jl_k + f_if_{j|h}l_k + f_{j|h}f_kl_i + f_jf_{k|h}l_i + f_{k|h}f_il_j + f_kf_{i|h}l_j) = 0 \end{aligned} \tag{14}$$

$$\begin{aligned} & \frac{f_{ijk|0} - L^2C^2|_i|_j|_{k|0}}{4C^2} - \frac{C^2_{|0}}{C^2}C_{ijk} - \frac{1}{2LC^2}(f_{ik|0}l_j + f_{jk|0}l_i + f_{ij|0}l_k) \\ & + \frac{1}{2L^2C^2}\pi_{(ijk)}(f_{i|0}h_{jk}) - \frac{1}{2L^2C^2}(f_{ij|0}f_k + f_{ij}f_{k|0} + f_{jk|0}f_i \\ & + f_{jk}f_{i|0} + f_{ik|0}f_j + f_{ik}f_{j|0}) - \frac{1}{2L^2C^2}(f_{i|0}l_jl_k + f_{j|0}l_il_k + f_{k|0}l_il_j) \\ & - \frac{1}{L^3C^2}(f_{i|0}f_jl_k + f_if_{j|0}l_k + f_{j|0}f_kl_i + f_jf_{k|0}l_i + f_{k|0}f_il_j + f_kf_{i|0}l_j) = 0 \end{aligned} \tag{15}$$

So, we have

Theorem 3.1 *If an n-dimensional Finsler space M^n satisfies the condition $L^2C^2 = f(y)+g(x)$, then the necessary and sufficient condition for M^n to be a Berwald space is that equation (14) holds good.*

Theorem 3.2 *If an n-dimensional Finsler space M^n satisfies the condition $L^2C^2 = f(y)+g(x)$, then the necessary and sufficient condition for M^n to be a Landsberg space is that equation (15) holds good.*

If h-covariant differentiation of f_i is vanishes then the theorem 3 and theorem 4 gives the result that the condition $C_{ijk|h} = 0$ (resp. $P_{ijk} = 0$) is equivalent to $C^2|_i|_j|_{k|h} = 0$ (resp. $C^2|_i|_j|_{k|0} = 0$). So, we have

Corollary 3.1 *If an n-dimensional Finsler space M^n satisfies the condition $L^2C^2 = f(y)+g(x)$, then the necessary and sufficient condition for M^n to be a Berwald space is that $C^2|_i|_j|_{k|h} = 0$ holds good.*

Corollary 3.2 *If an n-dimensional Finsler space M^n satisfies the condition $L^2C^2 = f(y)+g(x)$, then the necessary and sufficient condition for M^n to be a Berwald space is that $C^2|_i|_j|_{k|0} = 0$ holds good.*

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Bounds on Szeged and PI Indexes in terms of Second Zagreb Index

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Abstract: In this short note, we studied the *vertex version* and the *edge version* of the *Szeged index* and the *PI index* and obtained bounds for these indices in terms of the Second Zagreb index. Also, established the connections of bounds to the above sighted indices.

Key Words: Simple graph, Smarandache-Zagreb index, Szeged index, PI index, Zagreb index.

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§1. Introduction and Terminologies

Graph theory has provided chemists with a variety of useful tools, such as topological indices [9]. Let $G = (V, E)$ be a simple graph with $n = |V|$ vertices and $e = |E = E(G)|$ edges. For the vertices u, $v \in V$, the distance $d(u, v)$ is defined as the length of the shortest path between u and v in G. In theoretical Chemistry, molecular structure descriptors, the topological indices are used for modeling physic - chemical, toxicologic, biological and other properties of chemical compounds. Arguably, the best known of these indices is the *Wiener index* W [10], defined as the sum of the distances between all pairs of vertices of the graph G.

$$W(G) = \sum d(u, v).$$

The various extensions and generalization of the *Wiener index* are recently put forward.

Let $e = (u, v)$ be an edge of the graph G. The number of vertices of G whose distance to the vertex u is smaller than the distance to the vertex v is denoted by $n_u(e)$. Analogously, $n_v(e)$ is the number of vertices of G whose distance to the vertex v is smaller than the distance to the vertex u. Similarly $m_u(e)$ denotes the number of edges of G whose distance to the vertex u is smaller than the distance to the vertex v. The topological indices *vertex version* and the *edge version* of the *Szeged Index* and the *PI Index* [4,8] of G is defined as

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$$PI_v(G) = \sum[n_u(e) + n_v(e)]$$
$$PI_e(G) = \sum[m_u(e) + m_v(e)]$$
$$SZ_v(G) = \sum[n_u(e)n_v(e)]$$
$$SZ_e(G) = \sum[m_u(e)m_v(e)].$$

The structure-descriptor the *Zagreb index* [3,5,6-7], more precisely, the *first Zagreb index* is

$$M_1(G) = \sum d(u)^2$$

and the *second Zagreb index* is

$$M_2(G) = \sum d(u).d(v)$$

where $(u, v) \in E(G)$. Generally, let G be a graph and H its a subgraph. The *Smarandache-Zagreb index of G relative to H* is defined by

$$M^S(G) = \sum_{u \in V(H)} d^2(u) + \sum_{(u,v) \in E(G \setminus H)} d(u)d(v).$$

Particularly, if $H = G$ or $H = \emptyset$, we get the first or second Zagreb index $M_1(G)$ and $M_2(G)$, respectively.

The outline of the paper is as follows: Introduction and terminologies are described in the first section. In forthcoming section, we concentrate our efforts on initiate a systematic study on the vertex version and the edge version of the *Szeged index* and the *PI index* and obtained some bounds for these indices in terms of the *Second Zagreb index*. For other undefined notations and terminology from graph theory, the readers are referred to J.A. Bondy and et al [1].

§2. Relations of the Szeged Index and PI Index in Terms of Second Zagreb Index

In this section, we derived the relations connecting the Zagreb index on Compliment Graph of various graph operators with respect to the ladder graph, complete graph and wheel graph.

Theorem 2.1 *For a simple graph G with the first and the second Zagreb indices $M_1(G)$ and $M_2(G)$ respectively, then, $M_2(G) \leq \frac{1}{2}\sqrt{M_1(G)}$.*

Proof For an edge $(u, v) \in E(G)$,

$$[d(u) + d(v)]^2 \geq 4d(u)d(v)Rightarrow \sum[d(u) + d(v)]^2 \geq 4\sum d(u)d(v)].$$

Summing up similar inequalities of all the edges $e \in E(G)$ then,

$$4\sum d(u)d(v) \leq \sum[(d(u))^2]^{\frac{1}{2}} + \sum[(d(v))^2]^{\frac{1}{2}}$$
$$\Rightarrow 4M_2(G) \leq 2\sqrt{M_1(G)} \Rightarrow M_2(G) \leq \frac{1}{2}\sqrt{M_1(G)}.$$

This completes the proof. □

Remark The equality holds in the above relation only for the regular graph.

Theorem 2.2 *For a simple graph G with e edges and n vertices then,*

$$M_2(G) \leq SZ_v \leq en^2 + M_2(G)(1-n).$$

Proof For an edge $e = (u, v)\epsilon E(G)$, $n_u(e) \geq d(v)$ and $n_v(e) \geq d(v)$. Hence,

$$\begin{aligned}
&\Rightarrow d(u)d(v) \leq n_u(e)n_v(e) \\
&\Rightarrow \sum d(u)d(v) \leq \sum n_u(e)n_v(e) \\
&\Rightarrow M_2(G) \leq SZ_v(G). \qquad (2.1)
\end{aligned}$$

Also $n_u(e) \leq n - d(v)$ and $n_v(e) \leq n - d(u)$. From these

$$\begin{aligned}
&n_u(e)n_v(e) \leq n^2 - n[d(u) + d(v)] + d(u)d(v) \\
&\Rightarrow \sum [n_u(e)n_v(e)] \leq en^2 - n[d(u)d(v)] + d(u)d(v) \\
&\Rightarrow SZ_v(G) \leq en^2 + M_2(G)(1-n). \qquad (2.2)
\end{aligned}$$

From equations (2.1) and (2.2),

$$M_2(G) \leq SZ_v(G) \leq en^2 + M_2(G)(1-n). \qquad \Box$$

Theorem 2.3 *For a simple graph G with the vertex version of the PI index $PI_v(G)$ then,*

$$PI_v(G) \leq 2ne - M_2(G).$$

Proof We have

$$\begin{aligned}
&[n_u(e) + n_v(e)] \leq 2n - [d(u) + d(v) \\
&\Rightarrow \sum [n_u(e) + n_v(e)] \leq 2ne - \sum [d(u)d(v)] \\
&\Rightarrow PI_v(G) \leq 2ne - M_2(G). \qquad \Box
\end{aligned}$$

Theorem 2.4 *For a simple graph G with the edge version of the szeged index $SZ_e(G)$ then,*

$$SZ_e(G) \geq M_2(G).$$

Proof For any edge $e = (u, v) \in E(G)$, $m_u(e) \geq d(u) - 1$ and $m_v(e) \geq d(v) - 1$. Hence,

$$\begin{aligned}
&m_u(e)m_v(e) \geq [d(u) - 1][d(v) - 1] \\
&\Rightarrow m_u(e)m_v(e) \geq d(u)d(v) - [d(u) + d(v)] + 1 \\
&\Rightarrow \sum [m_u(e)m_v(e)] \geq \sum [d(u)d(v)] - \sum [d(u) + d(v)] + e \\
&\text{(where e is the number of edges)} \\
&\Rightarrow SZ_e(G) \geq M_2(G). \qquad \Box
\end{aligned}$$

Theorem 2.5 *For a graph G with the vertex version and the edge version of the PI index as $PI_v(G)$ and $PI_e(G)$ respectively, then*

$$PI_v(G) \geq PI_e(G) + 2e.$$

Proof For an edge $e = (u, v) \in E(G)$,

$$n_u(e) \geq m_u(e) + 1 \tag{2.3}$$

and

$$n_v(e) \geq m_v(e) + 1 \tag{2.4}$$

Hence,

$$\begin{aligned} &(n_u(e) + n_v(e)) \geq (m_u(e) + m_v(e)) + 2 \\ &\Rightarrow \sum(n_u(e) + n_v(e)) \geq \sum(m_u(e) + m_v(e)) + 2e \\ &\Rightarrow PI_v(G) \geq PI_e(G) + 2e. \end{aligned}$$

□

Theorem 2.6 *For a simple graph G then,*

$$SZ_v(G) \geq SZ_e(G) + PI_e(G) + e.$$

Proof From equations 2.3 and 2.4, we have,

$$n_u(e)n_v(e) \geq m_u(e)m_v(e) + [m_u(e) + m_v(e)] + 1.$$

Whence,

$$\begin{aligned} &\sum[n_u(e)n_v(e)] \geq \sum[m_u(e)m_v(e)] + \sum[m_u(e) + m_v(e)] + e \\ &\Rightarrow SZ_v(G) \geq SZ_e(G) + PI_e(G) + e. \end{aligned}$$

□

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Equations for Spacelike Biharmonic General Helices with Timelike Normal According to Bishop Frame in The Lorentzian Group of Rigid Motions $\mathbb{E}(1,1)$

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Abstract: In this paper, we study spacelike biharmonic general helices according to Bishop frame in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$. We characterize the spacelike biharmonic general helices in terms of their curvatures in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$.

Key Words: Biharmonic curve, bienergy, bitension field, bishop frame, rigid motion.

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§1. Introduction

A helix, sometimes also called a coil, is a curve for which the tangent makes a constant angle with a fixed line. The shortest path between two points on a cylinder (one not directly above the other) is a fractional turn of a helix, as can be seen by cutting the cylinder along one of its sides, flattening it out, and noting that a straight line connecting the points becomes helical upon re-wrapping. It is for this reason that squirrels chasing one another up and around tree trunks follow helical paths.

Helices can be either right-handed or left-handed. With the line of sight along the helix's axis, if a clockwise screwing motion moves the helix away from the observer, then it is called a right-handed helix; if towards the observer then it is a left-handed helix. Handedness (or chirality) is a property of the helix, not of the perspective: a right-handed helix cannot be turned or flipped to look like a left-handed one unless it is viewed in a mirror, and vice versa.

Most hardware screw threads are right-handed helices. The alpha helix in biology as well as the A and B forms of DNA are also right-handed helices. The Z form of DNA is left-handed.

The pitch of a helix is the width of one complete helix turn, measured parallel to the axis of the helix. A double helix consists of two (typically congruent) helices with the same axis, differing by a translation along the axis.

The notions of harmonic and biharmonic maps between Riemannian manifolds have been introduced by J. Eells and J.H. Sampson (see [4]).

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A smooth map $\phi : N \longrightarrow M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} |\mathcal{T}(\phi)|^2 \, dv_h,$$

where $\mathcal{T}(\phi) := \operatorname{tr}\nabla^{\phi} d\phi$ is the tension field of ϕ

The Euler–Lagrange equation of the bienergy is given by $\mathcal{T}_2(\phi) = 0$. Here the section $\mathcal{T}_2(\phi)$ is defined by

$$\mathcal{T}_2(\phi) = -\Delta_{\phi}\mathcal{T}(\phi) + \operatorname{tr}R\left(\mathcal{T}(\phi), d\phi\right) d\phi, \tag{1.1}$$

and called the bitension field of ϕ. Non-harmonic biharmonic maps are called proper biharmonic maps.

In this paper, we study spacelike biharmonic general helices according to Bishop frame in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$. We characterize the spacelike biharmonic general helices in terms of their curvatures in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$. Finally, we obtain parametric equations of spacelike biharmonic general helices according to Bishop frame in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$.

§2. **Preliminaries**

Let $\mathbb{E}(1,1)$ be the group of rigid motions of Euclidean 2-space. This consists of all matrices of the form

$$\begin{pmatrix} \cosh x & \sinh x & y \\ \sinh x & \cosh x & z \\ 0 & 0 & 1 \end{pmatrix}.$$

Topologically, $\mathbb{E}(1,1)$ is diffeomorphic to $\mathbb{R}^3$ under the map

$$\mathbb{E}(1,1) \longrightarrow \mathbb{R}^3 : \begin{pmatrix} \cosh x & \sinh x & y \\ \sinh x & \cosh x & z \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow (x, y, z),$$

It's Lie algebra has a basis consisting of

$$\mathbf{X}_1 = \frac{\partial}{\partial x},\ \mathbf{X}_2 = \cosh x \frac{\partial}{\partial y} + \sinh x \frac{\partial}{\partial z},\ \mathbf{X}_3 = \sinh x \frac{\partial}{\partial y} + \cosh x \frac{\partial}{\partial z},$$

for which

$$[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_3,\ \ [\mathbf{X}_2, \mathbf{X}_3] = 0,\ \ [\mathbf{X}_1, \mathbf{X}_3] = \mathbf{X}_2.$$

Put

$$x^1 = x,\ x^2 = \frac{1}{2}\left(y + z\right),\ x^3 = \frac{1}{2}\left(y - z\right).$$

Then, we get

$$\mathbf{X}_1 = \frac{\partial}{\partial x^1},\ \mathbf{X}_2 = \frac{1}{2}\left(e^{x^1}\frac{\partial}{\partial x^2} + e^{-x^1}\frac{\partial}{\partial x^3}\right),\ \mathbf{X}_3 = \frac{1}{2}\left(e^{x^1}\frac{\partial}{\partial x^2} - e^{-x^1}\frac{\partial}{\partial x^3}\right). \tag{2.1}$$

The bracket relations are

$$[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_3, \;\; [\mathbf{X}_2, \mathbf{X}_3] = 0, \;\; [\mathbf{X}_1, \mathbf{X}_3] = \mathbf{X}_2. \tag{2.2}$$

We consider left-invariant Lorentzian metrics which has a pseudo-orthonormal basis $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$. We consider left-invariant Lorentzian metric [10], given by

$$g = -\left(dx^1\right)^2 + \left(e^{-x^1}dx^2 + e^{x^1}dx^3\right)^2 + \left(e^{-x^1}dx^2 - e^{x^1}dx^3\right)^2, \tag{2.3}$$

where

$$g\left(\mathbf{X}_1, \mathbf{X}_1\right) = -1, \;\; g\left(\mathbf{X}_2, \mathbf{X}_2\right) = g\left(\mathbf{X}_3, \mathbf{X}_3\right) = 1. \tag{2.4}$$

Let coframe of our frame be defined by

$$\theta^1 = dx^1, \;\; \theta^2 = e^{-x^1}dx^2 + e^{x^1}dx^3, \;\; \theta^3 = e^{-x^1}dx^2 - e^{x^1}dx^3.$$

Proposition 2.1 *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above the following is true:*

$$\nabla = \begin{pmatrix} 0 & 0 & 0 \\ -\mathbf{X}_3 & 0 & -\mathbf{X}_1 \\ -\mathbf{X}_2 & -\mathbf{X}_1 & 0 \end{pmatrix}, \tag{2.5}$$

where the (i, j)-element in the table above equals $\nabla_{\mathbf{X}_i}\mathbf{X}_j$ for our basis

$$\{\mathbf{X}_k, k = 1, 2, 3\} = \{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}.$$

§3. Spacelike Biharmonic General Helices with Timelike Normal According to Bishop Frame in the Lorentzian Group of Rigid Motions $\mathbb{E}(1, 1)$

Let $\gamma : I \longrightarrow \mathbb{E}(1, 1)$ be a non geodesic spacelike curve on the $\mathbb{E}(1, 1)$ parametrized by arc length. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame fields tangent to the $\mathbb{E}(1, 1)$ along γ defined as follows:

$\mathbf{T}$ is the unit vector field γ' tangent to γ, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ), and $\mathbf{B}$ is chosen so that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= \kappa\mathbf{T} + \tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= \tau\mathbf{N}, \end{aligned}$$

where κ is the curvature of γ and τ is its torsion and

$$\begin{aligned} g\left(\mathbf{T}, \mathbf{T}\right) &= 1, \;\; g\left(\mathbf{N}, \mathbf{N}\right) = -1, \;\; g\left(\mathbf{B}, \mathbf{B}\right) = 1, \\ g\left(\mathbf{T}, \mathbf{N}\right) &= g\left(\mathbf{T}, \mathbf{B}\right) = g\left(\mathbf{N}, \mathbf{B}\right) = 0. \end{aligned}$$

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\begin{aligned}
&\nabla_{\mathbf{T}}\mathbf{T} = k_1\mathbf{M}_1 - k_2\mathbf{M}_2, \\
&\nabla_{\mathbf{T}}\mathbf{M}_1 = k_1\mathbf{T}, \\
&\nabla_{\mathbf{T}}\mathbf{M}_2 = k_2\mathbf{T},
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
&g(\mathbf{T},\mathbf{T}) = 1,\ g(\mathbf{M}_1,\mathbf{M}_1) = -1,\ g(\mathbf{M}_2,\mathbf{M}_2) = 1, \\
&g(\mathbf{T},\mathbf{M}_1) = g(\mathbf{T},\mathbf{M}_2) = g(\mathbf{M}_1,\mathbf{M}_2) = 0.
\end{aligned} \tag{3.2}$$

Here, we shall call the set $\{\mathbf{T},\mathbf{M}_1,\mathbf{M}_1\}$ as Bishop trihedra, k_1 and k_2 as Bishop curvatures and $\tau(s) = \psi'(s)$, $\kappa(s) = \sqrt{|k_2^2 - k_1^2|}$. Thus, Bishop curvatures are defined by

$$\begin{aligned}
k_1 &= \kappa(s)\sinh\psi(s), \\
k_2 &= \kappa(s)\cosh\psi(s).
\end{aligned}$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ we can write

$$\begin{aligned}
&\mathbf{T} = T^1\mathbf{e}_1 + T^2\mathbf{e}_2 + T^3\mathbf{e}_3, \\
&\mathbf{M}_1 = M_1^1\mathbf{e}_1 + M_1^2\mathbf{e}_2 + M_1^3\mathbf{e}_3, \\
&\mathbf{M}_2 = M_2^1\mathbf{e}_1 + M_2^2\mathbf{e}_2 + M_2^3\mathbf{e}_3.
\end{aligned} \tag{3.3}$$

Theorem 3.1 *$\gamma : I \longrightarrow \mathbb{E}(1,1)$ is a spacelike biharmonic curve with Bishop frame if and only if*

$$\begin{aligned}
&k_1^2 - k_2^2 = \text{constant} = C \neq 0, \\
&k_1'' + Ck_1 = -k_1\left[1 + 2\left(M_2^1\right)^2\right] + 2k_2M_1^1M_2^1, \\
&k_2'' + Ck_2 = -2k_1M_1^1M_2^1 - k_2\left[-1 + 2\left(M_1^1\right)^2\right].
\end{aligned} \tag{3.4}$$

Definition 3.2 *A regular spacelike curve $\gamma : I \longrightarrow \mathbb{E}(1,1)$ is called a general helix provided the spacelike unit vector $\mathbf{T}$ of the curve γ has constant angle θ with some fixed timelike unit vector u, that is*

$$g(\mathbf{T}(s), u) = \cosh\wp \text{ for all } s \in I. \tag{3.5}$$

Theorem 3.4 *Let $\gamma : I \longrightarrow \mathbb{E}(1,1)$ is a non geodesic spacelike biharmonic general helix with timelike normal in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$. Then, the parametric equations*

of γ *are*

$$\begin{aligned} x^1(s) &= \cosh\wp s + C_1, \\ x^2(s) &= \frac{\sqrt{1+\cosh^2\wp}\, e^{\cosh\wp s + C_1}}{2\cosh\wp}[\cos s + \sin s] + C_2, \\ x^3(s) &= \frac{\sqrt{1+\cosh^2\wp}\, e^{-\cosh\wp s - C_1}}{2\cosh\wp}[\cos s - \sin s] + C_3, \end{aligned} \tag{3.6}$$

where C_1, C_2, C_3 *are constants of integration.*

Proof Using (3.1) and (3.5) we have above system. □

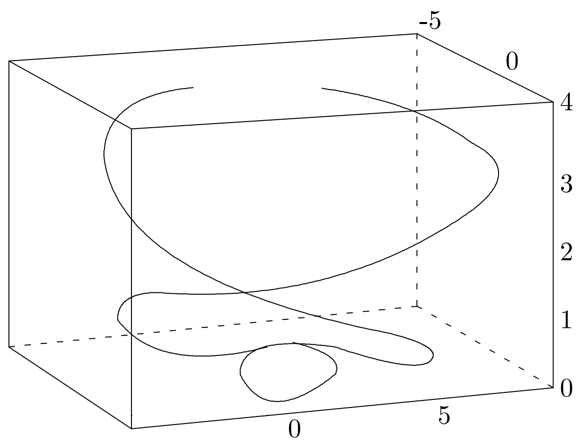

Fig.1

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Domination in Transformation Graph G^{+-+}

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Abstract: Let $G = (V, E)$ be a simple undirected graph of order n and size m. The transformation graph of G is a simple graph with vertex set $V(G) \cup E(G)$ in which adjacency is defined as follows: (a) two elements in $V(G)$ are adjacent if and only if they are adjacent in G (b) two elements in $E(G)$ are adjacent if and only if they are non-adjacent in G and (c) one element in $V(G)$ and one element in $E(G)$ are adjacent if and only if they are incident in G. It is denoted by G^{+-+}. A set $S \subseteq V(G)$ is a *dominating set* if every vertex in $V - S$ is adjacent to at least one vertex in S. The minimum cardinality taken over all dominating sets of G is called the *domination number* of G and is denoted by $\gamma(G)$. In this paper, we investigate the domination number of transformation graph. We determine the exact values for some standard graphs and obtain several bounds. Also we prove that for any connected graph G of order $n \geq 5$, $\gamma(G^{+-+}) \leq \lceil n/2 \rceil$.

Key Words: Transformation graph, domination number, Smarandachely k-dominating set, Smarandachely k-domination number.

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§1. Introduction

Let $G = (V, E)$ be a simple undirected graph of order n and size m. The *degree* of v in G is $|N(v)|$ and is denoted by $deg(v)$. The *maximum degree* of G is max $\{deg(v) : v \in V(G)\}$ and is denoted by $\Delta(G)$. The *minimum degree* of G is $\min\{deg(v) : v \in V(G)\}$ and is denoted by $\delta(G)$. A vertex v is said to be *pendant vertex* if $deg(v) = 1$. A vertex u is called *support* if u is adjacent to a pendant vertex.

A path on n vertices is denoted by P_n; a cycle on n vertices is denoted by C_n and a complete graph on n vertices is denoted by K_n. A complete bipartite graph in which one partite set has r vertices and another partite set has s vertices is denoted by $K_{r,s}$. The *corona* of two graphs G_1 and G_2, is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2. If G and H are any two graphs, then $G + H$ is the graph obtained from $G \cup H$ by joining each vertex of G to every

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vertex of H. $C_{n-1} + K_1$ is called the wheel on n vertices.

A set of vertices S in a graph G is said to be a *Smarandachely k-dominating set* if each vertex of G is dominated by at least k vertices of S and the *Smarandachely k-domination number* $\gamma_k(G)$ of G is the minimum cardinality of a Smarandachely k-dominating set of G. Particularly, if $k = 1$, such a set is called a dominating set of G and the Smarandachely 1-domination number of G is called the *domination number* of G and denoted by $\gamma(G)$ in general.

In [8], Paulraj Joseph and Arumugam studied domination parameters in subdivision graphs. Wallis [10] studied domination parameters of line graphs of designs and variations of Chess-board graph. The domination number of transformation graph G^{-+-} was studied in [1]. Terms not defined are used in the sense of [5].

The *transformation graph* of G is a simple graph with vertex set $V(G) \cup E(G)$ in which adjacency is defined as follows:(a) two elements in $V(G)$ are adjacent if and only if they are adjacent in G (b) two elements in $E(G)$ are adjacent if and only if they are non-adjacent in G and (c) one element in $V(G)$ and one element in $E(G)$ are adjacent if and only if they are incident in G. It is denoted by G^{+-+}. A graph G and it's transformation graph are given in Fig.1.1.

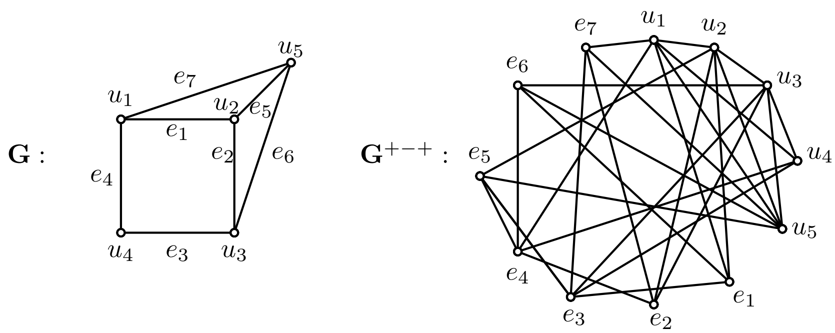


Fig.1.1 **A graph** G **and** G^{+-+}.

In this paper we study about domination number of the transformation graph G^{+-+}. We need the following theorems to obtain an upper bound for G^{+-+} in Section 5.

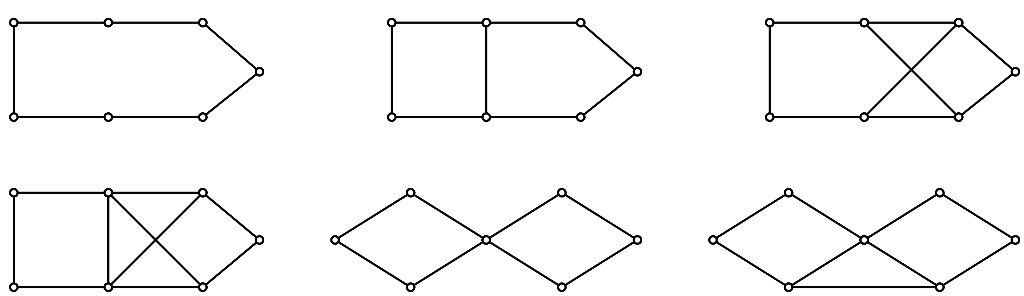


Fig.1.2 **Graphs in family** $\mathcal{A}$.

Theorem 1.1([7]) *If a graph G has no isolated vertices, then $\gamma(G) \leq n/2$.*

Theorem 1.2([3,4]) *For a graph G with even order n and no isolated vertices $\gamma(G) = n/2$ if and only if the components of G are the cycle C_4 or the corona $H \circ K_1$ for any connected graph H.*

Theorem 1.3([6]) *If G is a connected graph with $\delta(G) \geq 2$ and $G \notin \mathcal{A} \cup C_4$, then $\gamma(G) \leq 2n/5$.*

§2. Exact Values for Standard Graphs

In this section, we obtain the exact values for the domination number of the transformation graph where G is the star, complete graph, complete bipartite graph and wheel.

Theorem 2.1 $\gamma(G^{+-+}) = 1$ *if and only if* $G \cong K_{1,r},\ r \geq 1$.

Proof Assume that $G \cong K_{1,r},\ r \geq 1$. Let v be the full vertex of G. Then v is adjacent to all the vertices and all the edges of G in G^{+-+}. Therefore $D = \{v\}$ is a dominating set of G^{+-+}. Hence $\gamma(G^{+-+}) = 1$. Conversely, assume that $\gamma(G^{+-+}) = 1$. Let D be a minimum dominating set of G^{+-+}. If $D = \{e\}$, then e is incident with exactly two vertices of G and hence $G \cong K_2 \cong K_{1,1}$. If $D = \{v\}$, then v is adjacent to all the vertices and incident with all the edges of G. Hence $G \cong K_{1,r}$. □

Theorem 2.2 $\gamma(K_n^{+-+}) = 3,\ n \geq 4$.

Proof Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. By Theorem 2.1, $\gamma(K_n^{+-+}) \geq 2$. Let S be a subset of $V(K_{r,s}^{+-+})$.

Claim 1 No 2-element subset of $V(K_n^{+-+})$ is a dominating set of K_n^{+-+}.

Suppose $S = \{v_i, v_j\}$. Since $n \geq 4$, there exist two vertices v_k and v_r such that $v_kv_r \in E(G)$ which is adjacent to neither v_i nor v_j in K_n^{+-+}. Suppose $S = \{v_i, v_jv_k\}$. If $v_i = v_j$, then an edge which is incident with v_k is adjacent to neither v_i nor v_jv_k in K_n^{+-+}. If $v_i \neq v_j$, then since $n \geq 4$, there exists a vertex v_r such that $v_jv_r, v_kv_r \in E(G)$ are adjacent to neither v_i nor v_jv_k in K_n^{+-+}. Suppose $S = \{v_iv_j, v_kv_r\}$. If v_iv_j and v_kv_r are adjacent in K_n, then there exists a vertex $v_s \in V(K_n)$ which is adjacent to neither v_iv_j nor v_kv_r in K_n^{+-+}. If v_iv_j and v_kv_r are non-adjacent in K_n, then $v_iv_k,\ v_iv_r$ are adjacent to neither v_iv_j nor v_kv_r in K_n^{+-+}. Thus in all cases, S is not a dominating set of K_n^{+-+}.

Let v_i and v_j be any two vertices of K_n. Clearly v_i dominates all the vertices of K_n in K_n^{+-+}. All the edges which are adjacent to v_iv_j are dominated by the 2-element subset $\{v_i, v_j\}$ in K_n^{+-+} and all the remaining edges of K_n which are not adjacent to v_iv_j are dominated by v_iv_j in K_n^{+-+}. Hence $\{v_i, v_j, v_iv_j\}$ is a dominating set of K_n^{+-+}. Thus $\gamma(K_n^{+-+}) = 3$. □

Theorem 2.3 $\gamma(K_{r,s}^{+-+}) = 3$, *where* $r,\ s > 2$.

Proof Let $(X,\ Y)$ be the bipartition of $K_{r,s}$ with $|X| = r$ and $|Y| = s$. Since $r,\ s > 2$, by Theorem 2.1, $\gamma(K_{r,s}^{+-+}) \geq 2$. Let S be a subset of $V(K_{r,s}^{+-+})$.

Claim 1 No 2-element subset of $V(K_{r,s}^{+-+})$ is a dominating set of $K_{r,s}^{+-+}$.

Suppose $S = \{x, y\}$. If $x,\ y \in X$, then there exists a vertex $z \in X$ which is adjacent to neither x nor y in $K_{r,s}^{+-+}$. If $x \in X$ and $y \in Y$, then there exists an edge e whose end vertices are not in S. Then e is adjacent to neither x nor y in $K_{r,s}^{+-+}$. If x and y are edges of $K_{r,s}$, then at most four vertices of $K_{r,s}$ are dominated by S. Therefore, there exists a vertex of $K_{r,s}$ which is adjacent to neither x nor y in $K_{r,s}^{+-+}$. If $x \in X$ and $y \in E(K_{r,s})$, then at most two vertices of X are dominated by S in $K_{r,s}^{+-+}$. Therefore there exists a vertex $z \in X$ which is adjacent to neither x nor y in $K_{r,s}^{+-+}$. Thus in all cases, S is not a dominating set of $K_{r,s}^{+-+}$. Hence the claim.

Let $u \in X$ and $v \in Y$. Then all the vertices of $K_{r,s}$ and all the edges which are adjacent to uv are dominated by $\{u, v\}$ in $K_{r,s}^{+-+}$. Further, the remaining edges which are not adjacent to uv are dominated by uv. Hence $\{u, v, uv\}$ is a dominating set of $K_{r,s}^{+-+}$. Thus $\gamma(K_{r,s}^{+-+}) = 3$.□

Theorem 2.4 $\gamma(W_n^{+-+}) = 3,\ n \geq 4$.

Proof Let $v_1, v_2, \ldots, v_{n-1}$ be the vertices of degree 3 and $v$be the vertex of degree $n-1$ in W_n. Let $e_i = vv_i$ where $1 \leq i \leq n-1$, $e_{i(i+1)} = v_iv_{i+1}$ where $1 \leq i \leq n-2$ and $e_{1(n-1)} = v_1v_{n-1}$.

If $n = 4$, then $W_n \cong K_n$ and hence by Theorem 2.2, $\gamma(W_n^{+-+}) = 3$. Now, let $n \geq 5$. By Theorem 2.1, $\gamma(W_n^{+-+}) \geq 2$.

Claim 1 No 2-element subset of $V(W_n^{+-+})$ is a dominating set of W_n^{+-+}.

Now,$\{v, v_i\}$ is not a dominating set of W_n^{+-+} since there exists an edge e_{jk} which is not adjacent to either v or v_i in W_n^{+-+}; $\{v_i, v_j\}$ is not a dominating set of W_n^{+-+} since there exists an edge e_k which is not adjacent to either v_i or v_j in W_n^{+-+}. Hence no 2-element subset of $V(W_n)$ is a dominating set of W_n^{+-+}. Since any two edges of W_n are adjacent to at most 4 vertices of W_n in W_n^{+-+}, no 2-element subset of $E(W_n)$ is a dominating set of W_n^{+-+}. Now, $\{v, e_i\}$ is not a dominating set of W_n^{+-+} since adjacent edges of e_i in the cycle C_{n-1} are not adjacent to either v or e_i in W_n^{+-+}; $\{v, e_{ij}\}$ is not a dominating set of W_n^{+-+} since adjacent edges of e_{ij} in the cycle C_{n-1} are not adjacent to either v or e_{ij} in W_n^{+-+}; $\{v_i, e_j\}$ is not a dominating set of W_n^{+-+} since there exists an edge e_r which is not adjacent to either v_i or e_j; $\{v_i, e_{jk}\}$ is not a dominating set of W_n^{+-+} since at least one of $\{e_j, e_k\}$ is not adjacent to either v_i or e_{jk}. Hence no 2-element subset containing one vertex and one edge of W_n is not a dominating set of W_n^{+-+}. Thus the claim.

If $D' = \{v, e_{ij}, e_{jk}\}$, then in W_n^{+-+}, v is adjacent to all the vertices of W_n and all the spokes of W_n; e_{ij} is adjacent to all the non-adjacent edges of e_{ij}; e_{jk} is adjacent to all the adjacent edges of e_{ij} in the cycle C_{n-1}. Hence D' is a dominating set of W_n^{+-+}. Thus $\gamma(W_n^{+-+}) = 3$.□

§3. Some Bounds for $\gamma(G^{+-+})$

In this section, we obtain the upper bounds for the domination number of the transformation graph by considering the order, maximum and minimum degrees of G.

Theorem 3.1 *If G is a connected graph with $\Delta(G) = n - 2$, then $\gamma(G^{+-+}) \leq 3$.*

Proof Let v be a vertex of degree $\Delta(G)$ and $V - N[v] = \{u\}$. Let u be adjacent to $v_i \in N(v)$. Then v dominates all the vertices except u of G and all the edges incident with v in G^{+-+}. Also v_i dominates u and all the edges incident with v_i in G^{+-+}. Further, all the remaining edges are dominated by vv_i in G^{+-+}. Hence $\{v, v_i, vv_i\}$ is a dominating set of G^{+-+}. Thus $\gamma(G^{+-+}) \leq 3$. □

Theorem 3.2 *If a graph G has $diam(G) = 2$, then $\gamma(G^{+-+}) \leq \delta(G) + 1$ and the bound is sharp.*

Proof Let $v \in V(G)$ such that $deg(v) = \delta(G)$ and $N(v) = \{v_1, v_2, \ldots, v_\delta\}$. Then vv_i is adjacent to all the non-adjacent edges of vv_i in G^{+-+}. Further $N(v)$ dominates all the edges incident with v or v_i and also all the vertices of G in G^{+-+}. Hence $N(v) \cup \{vv_i\}$ is a dominating set of G^{+-+}. Thus $\gamma(G^{+-+}) \leq \delta(G) + 1$. Further, for the graph G in Fig.3.1, $\gamma(G^{+-+}) = 3 = \delta(G) + 1$. □

G :

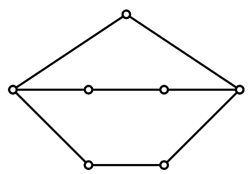


Fig.3.1 **A graph** G **with** $\gamma(G^{+-+}) = \delta(G) + 1$.

Theorem 3.3 *For any connected graph G with $\Delta(G) < n - 1$, $\gamma(G^{+-+}) \leq n - \Delta(G) + 1$ and the bound is sharp.*

Proof Let $deg(v) = \Delta(G)$ and $N(v) = \{v_1, v_2, \ldots, v_\Delta\}$. Since G is connected, there is a vertex $u \in V(G) - N[v]$ which is adjacent to at least one vertex $v_i \in N(v)$. Then $[(V(G) - N(v)) - \{u\}] \cup \{v_i\}$ dominates all the vertices of G. Now v dominates all the edges incident with v in G^{+-+}. The vertex v_i dominates all the edges incident with v_i and the edge vv_i dominates all the non-adjacent edges of vv_i in G^{+-+}. Therefore $[(V(G) - N(v)) - \{u\}] \cup \{v_i, vv_i\}$ is a dominating set of G^{+-+}. Hence $\gamma(G^{+-+}) \leq n - \Delta(G) + 1$. Further, for the graph G in Fig.3.2, $\gamma(G^{+-+}) = 4 = n - \Delta(G) + 1$. □

G :

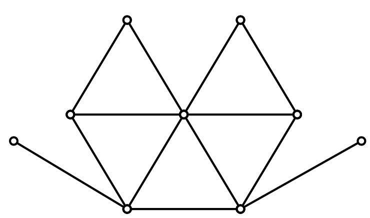


Fig.3.2 **A graph with** $\gamma(G^{+-+}) = n - \Delta(G) + 1$.

Theorem 3.4 *Let G be a connected graph of order $n > 2$ with $\Delta(G) = n - 1$ and v be a vertex*

of degree $\Delta(G)$. Then $\gamma(G^{+-+}) = 2$ if and only if $\langle N(v)\rangle$ is non-empty and it contains K_1 or K_2 or is isomorphic to $K_{1,r},\ r \geq 2$.

Proof Assume that $\gamma(G^{+-+}) = 2$. By Theorem 2.1, $\langle N(v)\rangle$ is non-empty. Suppose that $\langle N(v)\rangle$ is not isomorphic to $K_{1,r},\ r \geq 2$ and contains neither K_1 nor K_2. For $n \leq 4$, the result is easily verified.

Now, let $n \geq 5$. Then $\langle N(v)\rangle$ contains P_4 or C_3.

Since any 2-element subset of $E(G)$ is adjacent to at most 4 vertices of G in G^{+-+}, no 2-element subset of $E(G)$ is a dominating set of G^{+-+}. Let $S = \{x, y\}$. Suppose $x \in V(G)$ and $y \in E(G)$. If $x = v$, then there exists an edge in $\langle N(v)\rangle$ which is not adjacent to either x or y in G^{+-+}. If $x \in N(v)$, then there exists an edge which is incident with v which is not adjacent to either x or y in G^{+-+}. Suppose $x, y \in V(G)$. Since $n \geq 5$, there exist two vertices u and w of G such that $uw \in E(G)$ which is not adjacent to either x or y in G^{+-+}. Hence $\gamma(G^{+-+}) \geq 3$ which is a contradiction. Therefore $\langle N(v)\rangle$ contains K_1 or K_2 or is isomorphic to $K_{1,r},\ r \geq 2$.

Conversely, assume that $\langle N(v)\rangle$ is non-empty and it contains K_1 or K_2 or is isomorphic to $K_{1,r},\ r \geq 2$. By Theorem 2.1, $\gamma(G^{+-+}) \geq 2$. If $\langle N(v)\rangle$ contains $K_1 = \{u\}$, then $\{v, uv\}$ is a dominating set of G^{+-+}. If $\langle N(v)\rangle$ contains $K_2 = \{e\}$, then $\{v, e\}$ is a dominating set of G^{+-+}. If $\langle N(v)\rangle \cong K_{1,r},\ r \geq 2$ and u is the centre vertex, then $\{v, u\}$ is a dominating set of G^{+-+}. Hence $\gamma(G^{+-+}) = 2$. □

Remark 3.5 If $\Delta(G) = n - 1$, then $\gamma(G^{+-+})$ may be 3 which is greater than $n - \Delta(G) + 1$. For the graphs G_1 and G_2 in Fig.3.3, $\gamma(G_1^{+-+}) = \gamma(G_2^{+-+}) = 3$.

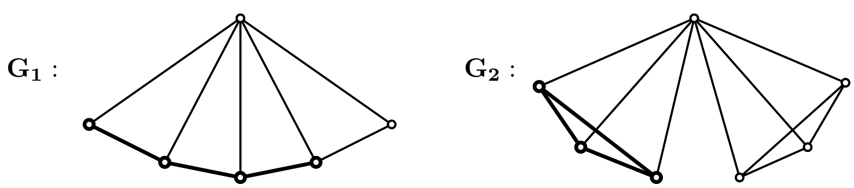


Fig.3.3 **Graphs with** $\gamma(G_1^{+-+}) = \gamma(G_2^{+-+}) > n - \Delta(G) + 1$.

Theorem 3.6 *For any graph G, $\gamma(G) \leq \gamma(G^{+-+}) \leq \gamma(G) + 2$ and the bounds are sharp.*

Proof Let D be a minimum dominating set of G and D' be a minimum dominating set of G^{+-+}.

Claim 1 $\gamma(G) \leq \gamma(G^{+-+})$.

Suppose $\gamma(G) > \gamma(G^{+-+})$. Then $|D| > |D'|$. If D' contains no edge of G, then D' is a dominating set of G with $|D'| < |D|$ which is a contradiction. If D' contains $k \geq 1$ edges of G, say $u_1v_1, u_2v_2, \ldots, u_kv_k$, then this k edges dominate $k_1 \leq 2k$ vertices of G in G^{+-+}. Therefore $[D' - \{u_1v_1, u_2v_2, \ldots, u_kv_k\}] \cup \{u_1, u_2, \ldots, u_k\}$ is a dominating set of cardinality $|D'| < |D|$ of G which is a contradiction.

Claim 2 $\gamma(G^{+-+}) \leq \gamma(G) + 2$.

All the vertices of G are dominated by D in G^{+-+}. If G is an empty graph, then $\gamma(G^{+-+}) =$

$\gamma(G) = n$. If G is non-empty, then G has an edge $e = uv$ whose one end is in D. Without loss of generality, let $u \in D$. Then every edge which is adjacent to e is adjacent to u or v in G^{+-+} and all the edges non-adjacent to e in G are adjacent to e in G^{+-+}. Hence $D \cup \{e, v\}$ is a dominating set of G^{+-+}. Thus $\gamma(G^{+-+}) \leq \gamma(G) + 2$.

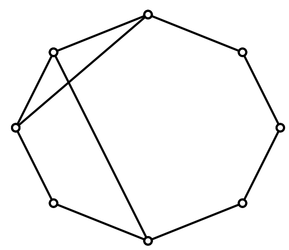
Further, $\gamma((H \circ K_1)^{+-+}) = n/2 = \gamma(H \circ K_1)$ and $\gamma((K_n)^{+-+}) = 3 = \gamma(K_n) + 2$, where $n \geq 4$. □

§4. Domination Transformation (DT) Classes

All the graphs are classified into three broad categories according to the lower and upper bounds in Theorem 3.6.

Definition 4.1 *A graph belongs to DT-class 1 if* $\gamma(G^{+-+}) = \gamma(G) + 1$.

$\mathbf{G_1}$:

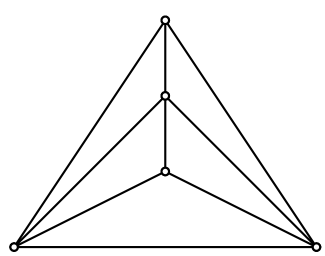


$\gamma(\mathbf{G_1}) = \mathbf{3}$; $\gamma(\mathbf{G_1^{+-+}}) = \mathbf{3+1=4}$

Fig.4.1 **A graph in DT-class 1.**

Definition 4.2 *A graph belongs to DT-class 2 if* $\gamma(G^{+-+}) = \gamma(G) + 2$.

$\mathbf{G_2}$:

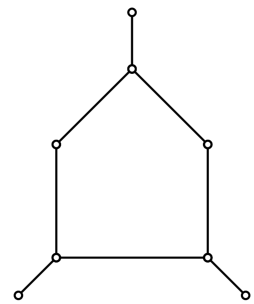


$\gamma(\mathbf{G_2}) = \mathbf{1}$; $\gamma(\mathbf{G_2^{+-+}}) = \mathbf{1+2=3}$

Fig.4.2 **A graph in DT-class 2.**

Definition 4.3 *A graph G belongs to DT-class 3 if* $\gamma(G^{+-+}) = \gamma(G)$.

G₃ :



$\gamma(\mathbf{G_3}) = \gamma(\mathbf{G_3^{+-+}}) = \mathbf{3}$

Fig.4.3 **A graph in DT-class 3.**

Theorem 4.4 *For $n \neq 4$, C_n is in DT-class 1.*

Proof We know that $\gamma(C_n) = \lceil n/3 \rceil$. We have to prove that $\gamma(C_n^{+-+}) = \lceil n/3 \rceil + 1$. By Theorem 3.6, $\gamma(C_n^{+-+}) \geq \lceil n/3 \rceil$. If $n = 3$, then $\gamma(C_n^{+-+}) = 2 = \lceil n/3 \rceil + 1$. Now let $n \geq 5$.

Claim 1 There is no dominating set with cardinality $\lceil n/3 \rceil$ in C_n^{+-+}.
Let S be any subset of $V(C_n^{+-+})$ with cardinality $\lceil n/3 \rceil$.

Case 1: $S \subseteq V(C_n)$.

If S is not a dominating set of C_n, then S is not a dominating set of C_n^{+-+}. Suppose S is a minimum dominating set of C_n. Since $n \geq 5$, there exist two adjacent vertices u and v which are not in S. Therefore, the edge $e = uv$ is not dominated by S. Hence S is not a dominating set of C_n^{+-+}.

Case 2 $S \subseteq E(C_n)$.

Since each edge of C_n is adjacent to exactly two vertices of C_n in C_n^{+-+}, at most $2|S| = 2\lceil n/3 \rceil$ vertices of C_n are dominated by S. Since $n \geq 5$, $2\lceil n/3 \rceil < n$. Therefore S is not a dominating set of C_n^{+-+}.

Case 3 $S \subseteq V(C_n) \cup E(C_n)$.

Let S contain $k \geq 1$ edges. Then at most $2k$ vertices of C_n are dominated by k edges of C_n in C_n^{+-+}. Further, at most $3(|S| - k)$ vertices of C_n are dominated by $|S| - k$ vertices of C_n. Now, $3|S| - 3k + 2k = 3\lceil n/3 \rceil - k = n - k < n$. Therefore, at most $n - k$ vertices of C_n are dominated by S. Hence S is not a dominating set of C_n^{+-+}. Thus in all cases, S is not a dominating set of C_n^{+-+}.

Let D be a minimum dominating set of C_n. Since $n \geq 5$, there exist two adjacent vertices x and y of C_n which are not in D and two vertices x' and y' which are adjacent to x and y respectively are in D. Then the edge $e = xy \in E(C_n)$is adjacent to all the edges whose end vertices are not in D. Further, two adjacent edges of e are adjacent to either x' or y' and all the edges whose one end in D are dominated by D in C_n^{+-+}. Hence $D \cup \{e\}$ is a dominating set of C_n^{+-+}. Thus $\gamma(C_n^{+-+}) = \lceil n/3 \rceil + 1$. $\square$

Theorem 4.5 *If G is a graph with $\delta(G) = 1$, then G is not in DT-class 2.*

Proof Let D be a minimum dominating set of G with all the supports. Let v be a pendant vertex and u be the support of v. Hence $u \in D$. Then all the vertices of G are dominated by D in G^{+-+}. Further, all the edges which are non-adjacent to the edge uv are adjacent to uv in G^{+-+} and all the edges which are adjacent to uv are adjacent to u in G^{+-+}. Hence $D \cup \{uv\}$ is a dominating set of G^{+-+}. Therefore, $\gamma(G^{+-+}) \leq \gamma(G) + 1$. Hence by Theorem 3.6, G is in DT-class 1 or DT-class 3. □

Theorem 4.6 *If $n \equiv 0, 2$ $(mod\ 3)$, $n \geq 6$, then P_n is in DT-class 1; otherwise DT-class 3.*

Proof The result can be easily verified for $n \leq 5$. Let $P_n = v_1 v_2 \ldots v_n$ and D be a minimum dominating set of P_n. By Theorem 3.6, $\gamma(P_n^{+-+}) \geq |D| = \lceil n/3 \rceil$.

Case 1 $n = 3k + 1$, $k \geq 2$.

Consider $D_1 = \{v_2, v_5, \ldots, v_{3k-1}\}$. Now, all the vertices of P_n except v_{3k+1} are dominated by D_1 in G^{+-+}. Also all edges whose one end vertex is in D_1 are dominated by D_1 in P_n^{+-+}. Further, $v_{3k}v_{3k+1}$ dominates v_{3k+1} and all the edges whose end vertices are not in D_1 at P_n^{+-+}. Hence $D_1 \cup \{v_{3k}v_{3k+1}\}$ is a dominating set of P_n^{+-+}. Therefore $\gamma(P_n^{+-+}) \leq |D_1| + 1 = \lceil n/3 \rceil$. Hence $\gamma(P_n^{+-+}) \leq \lceil n/3 \rceil$.

Case 2 $n = 3k$, $k \geq 2$.

Let S be any subset of $V(P_n^{+-+})$ with k elements.

Subcase 1 $S \subseteq V(P_n)$.

If S is not a dominating set of P_n, then S is not a dominating set of P_n^{+-+}. If S is a dominating set of P_n, then since $n = 3k$ and $n > 5$ there is an edge whose end vertices are not in S. Hence S is not a dominating set of P_n^{+-+}.

Subcase 2 $S \subseteq E(P_n)$.

Since each edge of P_n is adjacent to exactly two vertices of P_n in P_n^{+-+}, at most $2|S| = 2\lceil n/3 \rceil$ vertices of P_n are dominated by S. Since $n > 5$, $2\lceil n/3 \rceil < n$. Therefore, S is not a dominating set of G^{+-+}.

Subcase 3 $S \subseteq V(P_n) \cup E(P_n)$.

If S contains $r \geq 1$ edges, then at most $2r$ vertices are dominated by r edges. Further, at most $3(k-r)$ vertices are dominated by $|S| - r$ vertices in P_n^{+-+}. Now, $2r + 3(k-r) = 3k - r = n - r < n$.

Hence S is not a dominating set of P_n^{+-+}. By Theorem 4.5, $\gamma(P_n^{+-+}) \leq \lceil n/3 \rceil + 1$ and hence $\gamma(P_n^{+-+}) = \lceil n/3 \rceil + 1$.

Case 3 $n = 3k + 2$, $k \geq 2$.

Let $Q = (v_1 v_2 v_3 \ldots v_{3k} v_{3k+1})$ be a path on $3k+1$ vertices and D' be a minimum dominating set of Q^{+-+}. Then $|D'| = k + 1$.

Claim 1 D' contains no pendant vertex of Q.

Suppose $v_1 \in D'$. Then v_1 dominates v_2 and v_1v_2 in Q^{+-+}. By Case(ii), the remaining vertices and edges of Q are not dominated by any k-element subset of $V(Q^{+-+})$. Therefore, $|D'| > k+1$ which is a contradiction. Hence $v_1 \notin D'$. Similarly, $v_{3k+1} \notin D'$.

Therefore $v_{3k}v_{3k+1}$ or $v_{3k} \in D'$. Then v_{3k+2} is not dominated by D' in P_n^{+-+}. Hence $\gamma(P_n^{+-+}) > k+1$. But by Theorem 4.5, $\gamma(P_n^{+-+}) \leq k+2$. Thus $\gamma(P_n^{+-+}) = k+2 = \lceil n/3 \rceil + 1$. $\square$

Definition 4.7 *Two supports u and v are said to be successive supports if no internal vertex of any $u-v$ path is a support.*

Theorem 4.8 *Let T be a tree. If any two successive supports are of distance 1, 2 or 4 in T, then T is in DT-class 3.*

Proof Let D be a minimum dominating set of T containing all the supports of T. If any two successive supports are at distance 1, or 2, then D contains supports only. Then all the vertices and edges of T are dominated by D in T^{+-+}. Therefore $\gamma(T^{+-+}) \leq |D|$. By Theorem 3.6, $\gamma(T^{+-+}) = |D| = \gamma(T)$. If there exist two successive supports x and y at distance 4, then $x,\ y \in D$ and $w \in D$ where w is the vertex of distance 2 from x and y. Therefore, in T^{+-+} all the edges of $x-y$ path are dominated by $\{x, y, w\}$ which is also subset of D. Hence $\gamma(T^{+-+}) = \gamma(T)$. $\square$

Theorem 4.9 *Let T be a tree. If there exists a support v such that $d(v,x) \equiv 1(mod\ 3)$ for every successive support x of v, then T is in DT-class 3.*

Proof Let v be a support of T such that $d(v,x) \equiv 1(mod\ 3)$ for every successive support x of v. Let v' be the pendant vertex adjacent to v and x' be the pendant vertex adjacent to x. Let D be a minimum dominating set of G. Then v or $v' \in D$. Since $d(v,x) \equiv 1(mod\ 3)$, we can choose a minimum dominating set D_1 of T containing the neighbors of v such that $|D| = |D_1|$. Then $(D_1 - \{v'\}) \cup \{vv'\}$ dominates all the vertices of T in T^{+-+}. Further, vv' dominates all the edges which are non-adjacent to vv' and the adjacent edges of vv' are dominated by the neighbor of v in D_1. Hence $(D_1 - \{v'\}) \cup \{vv'\}$ is a dominating set of T^{+-+} of cardinality $|D| = \gamma(T)$. $\square$

Theorem 4.10 *If G is a disconnected graph with K_2 as one of the components of G, then G is in DT-class 3.*

Proof Let $G_1,\ G_2, \ldots, G_k$ be the components of G and $G_i \cong K_2 = uv$. Let $D_1,\ D_2, \ldots, D_i, \ldots, D_k$ be minimum dominating sets of $G_1, G_2, \ldots, G_i, \ldots, G_k$ respectively. Therefore, $|D_i| = 1$. All the vertices except u and v of G are dominated by $D_1 \cup D_2 \cup \ldots \cup D_{i-1} \cup D_{i+1} \cup \ldots \cup D_k$ in G^{+-+}. Further, $u,\ v$ and all the edges of G are dominated by uv in G^{+-+}. Hence $D_1 \cup D_2 \cup \ldots \cup D_{i-1} \cup D_{i+1} \cup \ldots \cup D_k \cup \{uv\}$ is a dominating set of cardinality $\gamma(G)$ of G^{+-+}. Hence by Theorem 3.6, $\gamma(G^{+-+}) = \gamma(G)$. $\square$

§5. The Upper Bound $\lceil n/2 \rceil$

In this section, we consider the connected graphs of order $6, 8, 10$ and prove that $\lceil n/2 \rceil$ is an upper bound for the domination number of the transformation graph where G is any connected graph of order n.

Lemma 5.1 *Let G be a connected graph of order 6. Then $\gamma(G^{+-+}) \leq 3$ and the bound is sharp.*

Proof If $\gamma(G) = 1$, then by Theorem 3.6, $\gamma(G^{+-+}) \leq 3$. If $\gamma(G) = 3$, then $G \cong H \circ K_1$ where $H \cong P_3$ or C_3. Then $V(H)$ is a minimum dominating set of G^{+-+}. Hence $\gamma(G^{+-+}) = 3$.

Now, let $\gamma(G) = 2$. If $\delta(G) = 1$, then by Theorem 4.5, $\gamma(G^{+-+}) \leq 3$. Assume that $\delta(G) \geq 2$. If $\Delta(G) = 5$ or 4, then by Theorem 3.1 and Theorem 3.6, $\gamma(G^{+-+}) \leq 3$. If $\Delta(G) = 2$, then $G \cong C_6$ and hence $\gamma(G^{+-+}) = 3$. Now, assume that $\Delta(G) = 3$. Let v be a vertex of degree 3. Then let $N(v) = \{v_1, v_2, v_3\}$ and $V - N[v] = \{u_1, u_2\}$. If $N(u_1) \cap N(u_2) \neq \phi$ in $N(v)$, then let $v_i \in N(u_1) \cap N(u_2)$. Now, v dominates all the vertices of $N[v]$ and all the edges which are incident with v in G^{+-+}. The vertex v_i dominates all the vertices of $V - N[v]$ and all the edges which are incident with v_i in G^{+-+}. Further vv_i dominates all the remaining edges which are not adjacent to vv_i. Hence $\{v, v_i, vv_i\}$ is a dominating set of G^{+-+}. If $N(u_1) \cap N(u_2) = \phi$ in $N(v)$, then let u_1 be adjacent to v_1 and u_2 be adjacent to v_2. Then u_1v_1 dominates u_1, v_1 and all the edges which are not adjacent to u_1v_1 in G^{+-+}; u_2v_2 dominates u_2, v_2 and all the edges which are adjacent to u_1v_1 except u_1u_2, v_1v_2 in G^{+-+}; vv_3 dominates v, v_3 and the edges u_1u_2, v_1v_2 in G^{+-+}. Hence $\{u_1v_1, u_2v_2, vv_3\}$ is a dominating set of G^{+-+}. Thus $\gamma(G^{+-+}) \leq 3$. Further, $\gamma(C_6^{+-+}) = 3$ and hence the bound is sharp. □

Lemma 5.2 *Let G be a connected graph of order 8. Then $\gamma(G^{+-+}) \leq 4$ and the bound is sharp.*

Proof If $\gamma(G) \leq 2$, then by Theorem 3.6, $\gamma(G^{+-+}) \leq 4$. If $\gamma(G) = 4$, then $G \cong H \circ K_1$ for some connected graph H of order 4. Then $V(H)$ is a minimum dominating set of G^{+-+}. Hence $\gamma(G^{+-+}) = 4$.

Now, let $\gamma(G) = 3$. If $\delta(G) = 1$, then by Theorem 4.5, $\gamma(G^{+-+}) \leq 4$. Assume that $\delta(G) \geq 2$. If $\Delta(G) = 6$ or 7, then by Theorem 3.1 and Theorem 3.6, $\gamma(G^{+-+}) \leq 3$. If $\Delta(G) = 2$, then $G \cong C_8$ and hence $\gamma(G^{+-+}) = 4$. Let v be a vertex of degree $\Delta(G)$. We consider the following three cases.

Case 1 $\Delta(G) = 5$.

Let $N(v) = \{v_1, v_2, v_3, v_4, v_5\}$ and $V - N[v] = \{u_1, u_2\}$. If $N(u_1) \cap N(u_2) \neq \phi$ in $N(v)$, then let $v_i \in N(u_1) \cap N(u_2)$. Now, v dominates all the vertices of $N[v]$ and all the edges which are incident with v in G^{+-+}. The vertex v_i dominates all the vertices of $V - N[v]$ and all the edges which are incident with v_i in G^{+-+}. Further vv_i dominates all the remaining edges which are not adjacent to vv_i. Hence $\{v, v_i, vv_i\}$ is a dominating set of G^{+-+}.

If $N(u_1) \cap N(u_2) = \phi$ in $N(v)$, then let u_1 be adjacent to v_1 and u_2 be adjacent to v_2. Let $D = \{v, v_1, vv_1, u_2\}$. Now, v dominates all the vertices of $N[v]$ and all the edges which are

incident with v in G^{+-+}; v_1 dominates all the edges which are incident with v_1 and the vertex u_1 in G^{+-+}; vv_1 dominates all the remaining edges which are not adjacent to vv_1 in G^{+-+}. Also u_2 dominates itself. Hence D is a dominating set of G^{+-+}. Thus $\gamma(G^{+-+}) \leq 4$.

Case 2 $\Delta(G) = 4$.

Let $N(v) = \{v_1, v_2, v_3, v_4\}$ and $V - N[v] = \{u_1, u_2, u_3\}$. If $N(u_i) \cap N(u_j) \neq \phi$ in $N(v)$, then let $v_k \in N(u_i) \cap N(u_j)$. Let $D = \{v, v_k, vv_k, u_r\}$ where $r \notin \{i, j\}$. All the vertices of G are dominated by D in G^{+-+}. Also all the adjacent edges of vv_k are dominated by $\{v, v_k\}$ and all the non-adjacent edges of vv_k are dominated by the edge vv_k in G^{+-+}. Hence D is a dominating set of G^{+-+}.

Now, let $N(u_i) \cap N(u_j) = \phi$ for all i and j in $N(v)$. Then the induced subgraph $\langle V - N[v] \rangle$ is not isomorphic to $\overline{K_3}$ and hence it is isomorphic to $K_2 \cup K_1$ or P_3 or C_3.

If $\langle V - N[v] \rangle \cong K_2 \cup K_1$, then let $u_1u_2 \in E(G)$ and $u_3v_i, u_3v_j \in E(G)$. Now, v dominates all the vertices of $N[v]$; u_1u_2 dominates u_1, u_2, all the edges in $N[v]$ and all the edges incident with u_3; u_3v_i dominates u_3 and all the edges adjacent to u_1u_2. Therefore $\{v, u_1u_2, u_3v_i\}$ is a dominating set of G^{+-+}. If $\langle V - N[v] \rangle \cong P_3$, then let u_1 and u_3 be pendant vertices in $\langle V - N[v] \rangle$ and $u_1v_i, u_3v_j \in E(G)$. v dominates all the vertices of $N[v]$ and all the edges which are incident with v in G^{+-+}. A vertex v_i dominates u_1 and all the edges incident with v_i in G^{+-+}. The edge vv_i dominates all the remaining edges which are non-adjacent to vv_i in G^{+-+}. Also, u_2 dominates u_3 and itself in G^{+-+}. Hence $\{v, v_i, vv_i, u_2\}$ is a dominating set of G^{+-+}. If $\langle V - N[v] \rangle \cong C_3$, then let u_1 be adjacent to v_i. Then $\{v, v_i, vv_i, u_2\}$ is a dominating set of G^{+-+}. Hence $\gamma(G^{+-+}) \leq 4$.

Case 3 $\Delta(G) = 3$.

Let $N(v) = \{v_1, v_2, v_3\}$ and $V - N[v] = \{u_1, u_2, u_3, u_4\}$. If $N(u_i) \cap N(u_j) \neq \phi$ for some $i,\ j$ in $N(v)$, then let $v_k \in N(u_1) \cap N(u_2)$. If $u_3u_4 \in E(G)$, then $\{v, v_k, vv_k, u_3u_4\}$ is a dominating set of G^{+-+}. If u_3 and u_4 are adjacent to a common vertex $x \in V(G)$, then $\{v, v_k, vv_k, x\}$ is a dominating set of G^{+-+}. If u_3 and u_4 are not adjacent to a common vertex and $u_3u_4 \notin E(G)$, then let u_3 be adjacent to v_i and u_1; and u_4 is adjacent to v_j and u_2. Hence $\{v, u_1, u_2, vv_k\}$ is a dominating set of G^{+-+}.

Now, let $N(u_i) \cap N(u_j) = \phi$ for all $i,\ j$ in $N(v)$. Then the induced subgraph $\langle V - N[v] \rangle$ is not isomorphic to $\overline{K_4}$ or $P_3 \cup K_1$ and hence it is isomorphic to $C_3 \cup K_1$ or P_4 or a graph with at least one vertex u_i of $V - N[v]$ is of degree three.

If $\langle V - N[v] \rangle \cong C_3 \cup K_1$ where $C_3 = u_1u_2u_3u_1$, then u_4 is adjacent to a vertex v_i. Therefore, $\{v, v_i, vv_i, u_1\}$ is a dominating set of G^{+-+}.

If $\langle V - N[v] \rangle \cong P_4 = u_1u_2u_3u_4$, then u_1 is adjacent to at least one vertex of v_i and u_4 is adjacent to at least one vertex of v_j. Let $D = \{v, u_2u_3, u_1v_i, u_4v_j\}$. The vertex v dominates all the vertices of $N[v]$ and all the edges incident with v in G^{+-+}. The edge u_2u_3 dominates $u_2,\ u_3$, all the edges in $N[v]$ and all the edges incident with u_1 or u_4 except u_1u_2 and u_3u_4 in G^{+-+}. Also the edge u_1v_i dominates $u_1,\ u_3u_4$ and all the edges incident with u_2 or u_3 except $u_1u_2,\ v_iu_2,\ v_iu_3$ in G^{+-+}. Further, u_4v_j dominates $u_4,\ u_1u_2,\ v_iu_2,\ v_iu_3$ in G^{+-+}. Hence D is a dominating set of G^{+-+}.

If at least one vertex u_i of $V - N[v]$ is of degree three in $\langle V - N[v] \rangle$, then at least one vertex

$u_j(\neq u_i)$ in $V - N[v]$ is adjacent to a vertex v_k of $N(v)$. Then $\{v, v_k, vv_k, u_i\}$ is a dominating set of G^{+-+}. Further, $\gamma(C_8^{+-+}) = 4$ and hence the bound is sharp. $\square$

Lemma 5.3 *Let G be a connected graph of order 10. Then $\gamma(G^{+-+}) \leq 5$. and the bound is sharp.*

Proof If $\gamma(G) \leq 3$, then by Theorem 3.6, $\gamma(G^{+-+}) \leq 5$. If $\gamma(G) = 5$, then $G = H \circ K_1$ for some connected graph H of order 5. Then $V(H)$ is a minimum dominating set of G^{+-+}. Hence $\gamma(G^{+-+}) = 5$. Now, let $\gamma(G) = 4$. If $\delta(G) = 1$, then by Theorem 4.5 $\gamma(G^{+-+}) \leq 5$. Now, assume that $\delta(G) \geq 2$. If $\Delta(G) = 8$ or 9, then by Theorem 3.1 and Theorem 3.6, $\gamma(G^{+-+}) \leq 3$. If $\Delta(G) = 2$, then $G \cong C_{10}$. Therefore $\gamma(G^{+-+}) = 5$. Let v be a vertex of degree $\Delta(G)$. Then we consider the following five cases.

Case 1 $\Delta(G) = 7$.

Let $N(v) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and $V - N[v] = \{u_1, u_2\}$. Therefore $\{v, u_1, u_2\}$ is a dominating set of G and hence $\gamma(G^{+-+}) \leq 3 + 2 = 5$.

Case 2 $\Delta(G) = 6$.

Let $N(v) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $V - N[v] = \{u_1, u_2, u_3\}$. If $N(u_i) \cap N(u_j) \neq \phi$ in $N(v)$, then let $v_k \in N(u_i) \cap N(u_j)$. Let $D' = \{v, v_k, vv_k, u_r\}$ where $r \notin \{i, j\}$. Clearly, all the vertices are dominated by D' in G^{+-+}. Also, all the adjacent edges of vv_k are dominated by $\{v, v_k\}$ and all the non-adjacent edges of vv_k are dominated by vv_k in G^{+-+}. Hence D' is a dominating set of G^{+-+}.

Assume that $N(u_i) \cap N(u_j) = \phi$ for all i, j in $N(v)$. Then the induced subgraph $\langle V - N[v] \rangle$ is isomorphic to $\overline{K_3}$ or $K_2 \cup K_1$ or P_3 or C_3.If $\langle V - N[v] \rangle \cong \overline{K_3}$, let u_i be adjacent to v_i. Then $\{v, v_i, vv_i, u_j, u_k\}$ is a dominating set of G^{+-+}. If $\langle V - N[v] \rangle \cong K_2 \cup K_1$, let $u_1u_2 \in E(G)$ and $u_3v_i, u_3v_j \in E(G)$. Then $\{v, u_1u_2, u_3v_i\}$ is a dominating set of G^{+-+}. If $\langle V - N[v] \rangle \cong P_3$ where $P_3 = u_1u_2u_3$, then $u_1v_i, u_3v_j \in E(G)$ and hence $\{v, v_i, vv_i, u_2\}$ is a dominating set of G^{+-+}. If $\langle V - N[v] \rangle \cong C_3$, let u_i be adjacent to v_i. Then $\{v, v_i, vv_i, u_j\}$ is a dominating set of G^{+-+}. Hence $\gamma(G^{+-+}) \leq 5$.

Case 3 $\Delta(G) = 5$.

Let $N(v) = \{v_1, v_2, v_3, v_4, v_5\}$ and $V - N[v] = \{u_1, u_2, u_3, u_4\}$. If $N(u_i) \cap N(u_j) \neq \phi$ in $N(v)$, let $v_k \in N(u_1) \cap N(u_2)$. Then $\{v, v_k, vv_k, u_3, u_4\}$ is a dominating set of G^{+-+}.

Now, let $N(u_i) \cap N(u_j) = \phi$ for all i and j in $N(v)$. Then the induced subgraph $\langle V - N[v] \rangle$ is not isomorphic to $\overline{K_4}$, or $K_2 \cup 2K_1$ and hence it is isomorphic to $P_3 \cup K_1$ or $C_3 \cup K_1$ or P_4 or C_4 or a graph with at least one vertex of $V - N[v]$ of degree 3. If $\langle V - N[v] \rangle \cong P_3 \cup K_1$ where $P_3 = u_1u_2u_3$, then u_1 is adjacent to at least one vertex v_k in $N(v)$ and hence $\{v, v_k, vv_k, u_2u_3, u_4\}$ is a dominating set of G^{+-+}. If $\langle V - N[v] \rangle \cong C_3 \cup K_1$ where $C_3 = u_1u_2u_3u_1$, then u_4 is adjacent to a vertex v_i. Therefore $\{v, v_i, vv_i, u_1\}$ is a dominating set of G^{+-+}. If $\langle V - N[v] \rangle \cong P_4 = u_1u_2u_3u_4$, then u_1 is adjacent to $v_i \in N(v)$ and u_4 is adjacent to v_j. Therefore $\{v, v_2v_3, u_1v_i, u_4v_j\}$ is a dominating set of G^{+-+}. If $\langle V - N[v] \rangle \cong C_4 = u_1u_2u_3u_4u_1$, let u_1 be adjacent to a vertex $v_j \in N(v)$. Then $\{v, v_j, vv_j, u_3\}$ is a dominating set of G^{+-+}. If at least one vertex u_i of $V - N[v]$ is of degree 3 in $\langle V - N[v] \rangle$, then at least one vertex u_j(may be u_i) in $V - N[v]$

is adjacent to a vertex $v_k \in N(v)$. Then $\{v, v_k, vv_k, u_i\}$ is a dominating set of G^{+-+}. Hence $\gamma(G^{+-+}) \leq 5$.

Case 4 $\Delta(G) = 4$.

Let $N(v) = \{v_1, v_2, v_3, v_4\}$ and $V - N[v] = \{u_1, u_2, u_3, u_4, u_5\}$. If $v_k \in N(u_i) \cap N(u_j) \cap N(u_k)$ for some i, j and k in $N(v)$, then $\{v, v_k, vv_k, u_r u_s\}$ is a dominating set of G^{+-+}. Now, let $N(u_i) \cap N(u_j) \cap N(u_k) = \phi$ for all i, j and k. Then the induced subgraph $\langle V - N[v] \rangle$ is non-empty and hence we consider two subcases.

Subcase 1 $\langle V - N[v] \rangle$ is disconnected.

If $\langle V - N[v] \rangle \cong K_2 \cup 3K_1$, let $u_1 u_2 \in E(G)$. Then each of u_3, u_4 and u_5 are adjacent to at least two vertices of $N(v)$ and each of u_1 and u_2 are adjacent to at least one vertex of $N(v)$. Hence $\{v_1, v_2, v_3, v_4, u_1 u_2\}$ is a dominating set of G^{+-+}. If $\langle V - N[v] \rangle \cong 2K_2 \cup K_1$, let $u_1 u_2, u_3 u_4 \in E(G)$ and u_5 be adjacent to v_i and v_j of $N(v)$. Then, $\{v, v_i, vv_i, u_1 u_2, u_3 u_4\}$ is a dominating set of G^{+-+}. If $\langle V - N[v] \rangle \cong P_3 \cup 2K_1$ where $P_3 = u_1 u_2 u_3$, let u_4 be adjacent to v_i and u_5 be adjacent to v_j. Then $\{v, v_i, vv_i, u_2, u_5\}$ is a dominating set of G^{+-+}. If $\langle V - N[v] \rangle \cong C_3 \cup 2K_1$ where $C_3 = u_1 u_2 u_3 u_1$, let u_4 be adjacent to v_i. Then $\{v, v_i, vv_i, u_2, u_5\}$ is a dominating set of G^{+-+}. If $\langle V - N[v] \rangle \cong P_4 \cup K_1$ or $C_4 \cup K_1$ where $P_4 = u_1 u_2 u_3 u_4$ and $C_4 = u_1 u_2 u_3 u_4 u_1$, let u_5 be adjacent to v_i. Then, $\{v, v_i, vv_i, u_1 u_2, u_3 u_4\}$ is a dominating set of G^{+-+}. If $\langle V - N[v] \rangle$ has exactly one isolated vertex u_i and a vertex u_j of degree 3, let u_k (or u_j) be adjacent to v_k and u_i be adjacent to v_r. Then $\{v, v_r, vv_r, u_j\}$ is a dominating set of G^{+-+}.

Subcase 2 $\langle V - N[v] \rangle$ is connected.

If $\langle V - N[v] \rangle \cong P_5$ where $P_5 = u_1 u_2 u_3 u_4 u_5$, then u_1 and u_5 must be adjacent to v_i and v_j respectively. Therefore $\{v, v_i, vv_i, u_2, u_4\}$ is a dominating set of G^{+-+}. If $\langle V - N[v] \rangle \cong C_5$, where $C_5 = u_1 u_2 u_3 u_4 u_5 u_1$, then $u_i v_j \in E(G)$. Then $\{v, v_j, vv_j, u_2, u_4\}$ is a dominating set of G^{+-+}. If $\langle V - N[v] \rangle$ has a vertex u_i of degree 3 and no isolated vertex, then there exists $u_j \in V - N[v]$ such that $u_i u_j \notin E(G)$ and there is u_k (may be u_i or u_j) which is adjacent to v_k. Therefore $\{v, v_k, vv_k, u_i, u_j\}$ is a dominating set of G^{+-+}. If $\langle V - N[v] \rangle$ has a vertex u_i of degree 4, then a vertex $u_j (\neq u_i)$ is adjacent to v_j and hence $\{v, v_j, vv_j, u_j\}$ is a dominating set of G^{+-+}. Thus $\gamma(G^{+-+}) \leq 5$.

Case 5 $\Delta(G) = 3$.

Let $N(v) = \{v_1, v_2, v_3\}$ and $V - N[v] = \{u_1, u_2, u_3, u_4, u_5, u_6\}$. Then $\langle V - N[v] \rangle$ has at most two isolated vertices and $\langle V - N[v] \rangle$ is not isomorphic to $2K_1 \cup 2K_2$. Hence it is isomorphic to $P_4 \cup 2K_1$ or $C_4 \cup 2K_1$ or $P_5 \cup K_1$ or $C_5 \cup K_1$ or $3K_2$ or P_6 or C_6 or a graph with a vertex of degree 3.

If $\langle V - N[v] \rangle \cong P_4 \cup 2K_1$ where $P_4 = u_1 u_2 u_3 u_4$, let $u_1 v_1 \in E(G)$. Then $\{v_1, v_2, v_3, u_3, vv_1\}$ is a dominating set of G^{+-+}. If $\langle V - N[v] \rangle \cong C_4 \cup 2K_1$ where $C_4 = u_1 u_2 u_3 u_4 u_1$, let $u_i v_j \in E(G)$. Then $\{v_1, v_2, v_3, vv_j, u_3\}$ is a dominating set of G^{+-+}. If $\langle V - N[v] \rangle$ has two isolated vertices and a vertex u_i of degree 3, then $\{v_1, v_2, v_3, vv_1, u_i\}$ is a dominating set of G^{+-+}. If $\langle V - N[v] \rangle \cong P_5 \cup K_1$, or $C_5 \cup K_1$ where $P_5 = u_1 u_2 u_3 u_4 u_5$ and $C_5 = u_1 u_2 u_3 u_4 u_5 u_1$, let u_5 be adjacent to v_i. Then $\{v, v_i, vv_i, u_1, u_4\}$ is a dominating set of G^{+-+}. If $\langle V - N[v] \rangle$

has exactly one isolated vertex u_i and a vertex u_j of degree 3, let $u_ju_k \notin E(G)$ and u_i be adjacent to v_i. Then $\{v, v_i, vv_i, u_j, u_k\}$ is a dominating set of G^{+-+}. If $\langle V - N[v]\rangle \cong 3K_2$ and $u_1u_2, u_3u_4, u_5u_6 \in E(G)$, let $u_1v_i \in E(G)$. Then $\{v, v_i, vv_i, u_3, u_5\}$ is a dominating set of G^{+-+}. If $\langle V - N[v]\rangle \cong P_6$ or C_6 where $P_6 = u_1u_2u_3u_4u_5u_6$ and $C_6 = u_1u_2u_3u_4u_5u_6u_1$, then $\{u_2, u_5, v, v_i, vv_i\}$ is a dominating set of G^{+-+}. If $\langle V - N[v]\rangle$ has a vertex u_i of degree 3 and $u_iu_j, u_iu_k \notin E(G)$, then u_j is adjacent to a vertex v_j of $N(v)$. Therefore $\{v, v_j, vv_j, u_i, u_k\}$ is a dominating set of G^{+-+}. Hence $\gamma(G^{+-+}) \leq 5$. Further, $\gamma(C_{10}^{+-+}) = 5$ and hence the bound is sharp. □

Theorem 5.4 *Let G be a connected graph of order $n \geq 5$. Then $\gamma(G^{+-+}) \leq \lceil n/2 \rceil$ and the bound is sharp.*

Proof If for $k \geq 2$, $\gamma(G) \leq \lceil n/2 \rceil - k$, then by Theorem 3.6, $\gamma(G^{+-+}) \leq \lceil n/2 \rceil$. Hence it is enough if we consider the following two cases.

Case 1 $\gamma(G) = \lceil n/2 \rceil$.

By Theorem 1.1, $\gamma(G) \leq n/2$ and hence n is even. Therefore, by Theorem 1.2, $G \cong C_4$ or $H \circ K_1$ for some connected graph H. If $G \cong C_4$, then $\gamma(G^{+-+}) = \gamma(G) = n/2$. If $G \cong H \circ K_1$, then all the supports of G from a minimum dominating set D of G. Since each edge of G is incident with at least one vertex of D, D is also a dominating set of G^{+-+}. Therefore, $\gamma(G^{+-+}) \leq n/2$.

Case 2 $\gamma(G) = \lceil n/2 \rceil - 1$.

If $\delta(G) = 1$, then by Theorem 4.5, $\gamma(G^{+-+}) \leq \gamma(G) + 1 = \lceil n/2 \rceil$.

Now, let $\delta(G) \geq 2$. If $G \in \mathcal{A}$, then $\gamma(G^{+-+}) = 3 < \lceil n/2 \rceil$. If $G \notin \mathcal{A}$, then by Theorem 1.3, $\gamma(G) \leq 2n/5$.

Subcase 1 n is odd.

Then $\gamma(G) = \lceil n/2 \rceil - 1 \leq 2n/5$. Therefore, $n \leq 5$. Hence $n = 5$. Clearly for all connected graphs on 5 vertices, $\gamma(G^{+-+}) \leq 3 = \lceil n/2 \rceil$.

Subcase 2 n is even.

Then $n/2 - 1 \leq 2n/5$. Thus $n \leq 10$. If $n = 6$, then by Lemma 5.1, $\gamma(G^{+-+}) \leq 3$. If $n = 8$, then by Lemma 5.2, $\gamma(G^{+-+}) \leq 4$. If $n = 10$, then by Lemma 5.3, $\gamma(G^{+-+}) \leq 5$. Further, for the graph G given in Fig.5.1, $\gamma(G^{+-+}) = 3 = \lceil n/2 \rceil$ and hence the bound is sharp. □

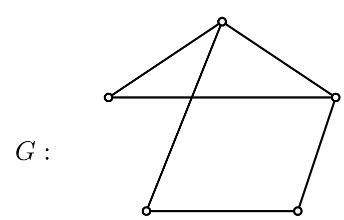


Fig.5.1 **A graph G with $G^{+-+} = \lceil n/2 \rceil$.**

Open Problems

We present open problems following:

1. Characterize the graphs which attain the bound given in Theorem 3.3.
2. Characterize the extremal graphs in Theorem 5.4.

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Combinatorial Aspects of a Measure of Rank Correlation Due to Kendall and its Relation to Complete Signed Digraphs

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Abstract: In this paper, we shall present an account of certain combinatorial aspects of a measure of rank correlation due to Kendall (1938) and point out its relation to the analysis of patterns of preference and indifference which in recent years, have been matters of intense discussion among the social psychologists because of their fundamental role in dealing with certain vital issues of social decision theory.

Key Words: Smarandachely k-signed graph, Smarandachely k-marked graph, signed digraph, rank, Kendalls τ.

AMS(2010): 05C22

§1. Introduction

For standard terminology and notion in digraph theory, we refer the reader to the classic textbooks of Bondy and Murty [1]and Harary et al. [3]; the non-standard will be given in this paper as and when required.

A *Smarandachely k-signed graph* (*Smarandachely k-marked graph*) is an ordered pair $S = (G, \sigma)$ $(S = (G, \mu))$ where $G = (V, E)$ is a graph called *underlying graph of* S and $\sigma : E \rightarrow (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$ $(\mu : V \rightarrow (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k))$ is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-signed graph or Smarandachely 2-marked graph is called a *signed graph* or a *marked graph*. A signed digraph $S = (D, \sigma)$ is *balanced*, if every semicycle of S is positive (See [3]). Equivalently, a signed digraph is balanced if every semicycle has an even number of negative arcs. The following characterization of balanced signed digraphs is obtained in [6].

Proposition 1.1(E. Sampathkumar et al. [6]) *A signed digraph $S = (D, \sigma)$ is balanced if, and only if, there exist a marking μ of its vertices such that each arc $\overrightarrow{uv}$ in S satisfies $\sigma(\overrightarrow{uv}) = \mu(u)\mu(v)$.*

In [6], the authors also introduced the switching and cycle isomorphism for signed digraphs. The *rank* means the position of an item or datum in relation to others which have been arranged according to some specific criterion, when used as verb, it means the act of assigning a rank to

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each term or datum according to the specified criterion (See, Wolman [7]). Ranking is then the arrangement of a series of values, scores or individuals in the order of their magnitude which may be decreasing or increasing. The problem of ranking individuals according to the extent of a prescribed attribute possessed or exhibited by them is of primary importance in the process of interpretation of statistical data for decision making in a wide variety of situations arising in experimental behavioral sciences. Moreover, when the individuals are ranked separately with regard to two different attributes, the two rankings may not be the same in general and hence the problem of effectively comparing the two rankings arises. A natural approach to such comparison is to quantify the content of correlation between the two rankings in some way that would reflect the individuals standing in each of the two rankings. Such a numerical value used to represent the content of correlation that may exist between any two given rankings is called a *measure of rank correlation*.

Several types of rank correlation measures exist in literature (See, Guilford and Fruchter [2], Kendall [4]). Of particular interest to us in this paper is Kendall τ ('tau') measure which resets on no regression analytic assumptions (See, Kendall [4]). This measure has numerous applications, including the testing of hypotheses, but bears no direct relation to the traditional family of productmoment correlations (See, Guilford and Fruchtter [2]).

§2. Complete Signed Digraphs and of Measure of Rank Correlation

Given two rankings A and B of n individuals $v_1, v_2, ..., v_n$, Kendall [4] has defined a new measure τ of rank correlation between A and B. We give here a method of construction of a complete signed digraph $S = (D, \sigma)$ with n vertices, from which we can easily find Kendall s τ by the formula $\tau = \frac{P-N}{P+N}$, where P and N respectively denote the number of positive arcs and number of negative arcs in S. A complete signed digraph is a complete digraph in which every arc is assigned either $+$ or $-$.

In [5], Sampathkumar and Nanjundaswamy obtained Kendall s τ for complete signed graphs. By the motivation of Kendalls τ for complete signed graphs, here we make an attempt to obtain the same for complete signed digraphs. Let $V = v_1, v_2, ..., v_n$ and a ranking of V is a bijective map $A : V \to \{1, 2, ..., n\}$.

Let A and B be two rankings of $V = \{v_1, v_2, ..., v_n\}$. Construct a signed digraph S on the complete digraph $\overrightarrow{K_n}$ whose vertices are labeled $v_1, v_2, ..., v_n$ as follows: consider A as the objective ranking and change the label of $\overrightarrow{K_n}$ according to the rule: $v_i \to V_{A(v_i)}$, for all $i \in \{1, 2, ..., n\}$. Observe that $B(V_j) = B(v_j)$, whenever $A(v_i) = j$. Now, label the arcs of $\overrightarrow{K_n}$ with respect to the new labeling recursively as below. For each $i = 1, 2, ..., n-1$,

$$\sigma(\overrightarrow{V_iV_j}) = \begin{cases} +, & \text{if } B(V_i) < B(V_j) \\ -, & \text{otherwise} \end{cases}$$

for each j, $j = i+1, i+2, ..., n$. From the above, we can easily observe that

$$\sigma(\overrightarrow{V_jV_i}) = \begin{cases} +, & \text{if } B(V_j) < B(V_i) \\ -, & \text{otherwise} \end{cases}$$

Hence, let P and N denote the number of positive and negative arcs respectively in the signed digraph S so obtained. Then the Kendalls measure τ of rank correlation is given by $\tau = \dfrac{P-N}{P+N}$. We now illustrate the above method of finding τ by an example.

Example 2.1 Let A and B be the two rankings of four individuals v_1, v_2, v_3 and v_4 given as follows:

Individuals	v_1	v_2	v_3	v_4
Ranking A :	3	1	4	2
Ranking B :	1	2	3	4

Draw the complete signed digraph S on v_1, v_2, v_3 and v_4. Consider A as the objective ranking. Relabel the vertices of S as follows: $v_1 \to V_3$, $v_2 \to V_1$, $v_3 \to V_4$ and $v_4 \to V_2$.

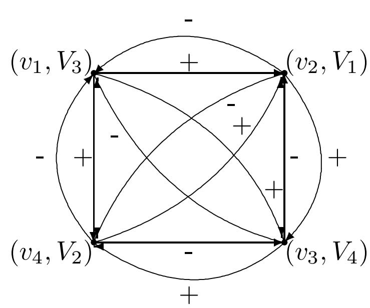

Fig.1

In Fig.1, $B(V_3) = B(v_1) = 1$, $B(V_1) = B(v_2) = 2$, $B(V_4) = B(v_3) = 3$ and $B(V_2) = B(v_4) = 4$. Here, we sign the arcs of D as mentioned above. For example, since $B(V_1) > B(V_3)$ $(B(V_3) < B(V_1))$ the arc $e = \overrightarrow{V_1V_2}$ $(e = \overrightarrow{V_2V_1})$ is labeled as negative (positive).

The signed digraph obtained in the above figure has $P = 6$ and $N = 6$. Hence, $\tau = 0$.

We can easily deduce the following properties of τ from the formula $\tau = \dfrac{P-N}{P+N}$:

1. $|\tau| \leq 1$

2. There is a perfect positive correlation between the rankings of A and B $\Leftrightarrow$ all arcs in S are positive, i.e, $N = 0 \Leftrightarrow \tau = 1$.

3. The rankings A and B are exactly inverted $\Leftrightarrow$ all arcs are negative, i.e, $P = 0 \Leftrightarrow \tau = -1$.

4. If the arc $\overrightarrow{V_1V_n}$ in the signed digraph is positive (negative) then it follows by the construction of that $P \geq N$ $(P \leq N)$ and hence $\tau \geq 0$ $(\tau \leq 0)$.

5. Since each arc of the signed digraph S is labeled either positive or negative, the probability that an arc is positive (negative) is $\frac{1}{2}$, and hence the number of positive (negative) arcs is a binomial variate. Since the probability that an arc is positive is exactly equal to the probability that is negative, the limiting case of distribution of number of individuals, is

increased indefinitely, is the normal distribution. Hence, the distribution of τ tends to normality for large n.

The original Kendalls method of finding τ goes as follows: Rearrange A as an objective order A' and write below it corresponding ranks in B to obtain the relative permutation $B' = (b'_1, b'_2, ..., b'_n)$. For each j, $j = 1, 2, ..., n-1$, consider the order of allot $n-j$ ordered pairs (b'_j, b'_k), $k = j+1, ..., n$ and allot score $+1$ to (b'_j, b'_k) if $b'_j < b'_k$ (then we say that (b'_j, b'_k) is in the correct order) and allot a score -1 to (b'_j, b'_k) if $b'_j > b'_k$ (then we say that (b'_j, b'_k) is in the wrong order). The sum of all these scores is called the *actual score*. The actual score of the objective order $A' = (1, 2, ..., n)$ is thus the combinatory function n_{C_2} and obviously must be the maximum possible score for any ranking of n items. Kendalls definition of τ is then,

$$\tau = \frac{\text{actualscore}}{\text{maximumscore}}.$$

Applying this procedure to the rankings A and B of above Example, we find and then computing τ according to $\tau = \frac{\text{actualscore}}{\text{maximumscore}}$ it turns out that $\tau = 0$, the value which also came out by computing according to $\tau = \frac{P-N}{P+N}$. That in general $\tau = \frac{P-N}{P+N}$ and $\tau = \frac{\text{actualscore}}{\text{maximumscore}}$ are equivalent.

Individuals	v_1	v_2	v_3	v_4
Ranking A :	1	2	3	4
Ranking B :	2	4	1	3

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Laplacian Energy of Certain Graphs

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Abstract: Let G be a graph with n vertices and m edges. Let $\mu_1, \mu_2, \cdots, \mu_n$ be the eigenvalues of the Laplacian matrix of G. The Laplacian energy $LE(G) = \sum\limits_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$. In this paper, we calculate the exact Laplacian energy of complete graph, complete bipartite graph, path, cycle and friendship graph.

Key Words: Complete graph, complete bipartite graph, path, cycle, friendship graph.

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§1. Introduction

Throughout this paper, by a graph we mean a finite, undirected, simple graph G with n vertices and m edges. Let d_i be the degree of the i^{th} vertex of $G, i = 1, 2, \cdots, n$.

Definition 1.1([3]) *Let $A(G) = [a_{ij}]$ be the $(0,1)$ adjacency matrix, $D(G) = diag(d_1, d_2, \cdots, d_n)$, the diagonal matrix with vertex degrees $d_1, d_2, \cdots, d_n$ of its vertices $v_1, v_2, \cdots, v_n$ of a graph G. Then $L(G) = D(G) - A(G)$ is called the Laplacian matrix of the graph G.*

It is symmetric, singular and positive semi - definite. All its eigenvalues $\mu_1, \mu_2, \cdots, \mu_n$ are real and nonnegative and form the Laplacian spectrum. It is well known that one of the eigen values is zero.

Definition 1.2([3]) *If G is a graph with n vertices and m edges, and its Laplacian eigen values are $\mu_1, \mu_2, \cdots, \mu_n$ then the Laplacian energy of G , denoted by $LE(G)$, is $\sum\limits_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$. i.e.,* $LE(G) = \sum\limits_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$.

This quantity has a long known chemical application for details see the surveys [1,4,5]. If the graph G has one vertex then the Laplacian energy is zero.

Property 1.3([3])

(1) $LE(G) \leq \sqrt{2Mn}$;

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(2) $LE(G) \leq \dfrac{2m}{n} + \sqrt{(n-1)\left[2M - (\dfrac{2m}{n}\right)^2\Big]}$;

(3) $2\sqrt{M} \leq LE(G) \leq 2M$, *where* $M = m + \dfrac{1}{2}\sum_{i=1}^{n}\left(d_i - \dfrac{2m}{n}\right)^2$.

§2. The Laplacian Energy of Complete Graphs

Definition 2.1([2]) *A simple graph in which each pair of distinct vertices is joined by an edge is called a complete graph.*

Theorem 2.2 *The Laplacian energy of the complete graph K_n on n vertices is* $2(n-1)$.

Proof The eigenvalues of the Laplacian matrix of the complete graph K_n on n vertices and $\dfrac{n(n-1)}{2}$ edges are $\mu_1 = 0$ and multiplicity of the eigen values n as $n-1$, i.e., $\mu_1 = 0, \mu_2 = \mu_3 = \cdots = \mu_n = n$. Thus

$$LE(K_n) = \sum_{i=1}^{n} |(\mu_i - (n-1)| = |0-(n-1)| + (n-1)|n-(n-1)| = 2(n-1). \qquad \square$$

§3. The Laplacian Energy of Complete Bipartite Graphs

Definition 3.1([2]) *A bipartite graph is one whose vertex set can be partitioned into two subsets X and Y, so that each edge has one end in X and one end in Y; such a partition(X,Y) is called a bipartition of the graph.*

Definition 3.2([2]) *A complete bipartite graph is a simple bipartite graph with bipartition (X,Y) in which each vertex of X is joined to each vertex of Y; if $|X| = m$ and $|Y| = n$, such a graph is denoted by $K_{m,n}$.*

Definition 3.3([6]) *The Star graph $K_{1,n}$ is a tree on n+1 vertices with one vertex having degree n and the other n vertices having degree* 1.

Theorem 3.4 *The Laplacian energy of the complete bipartite graph $K_{m,n}$ with $m+n$ vertices and mn edges is*

$$\frac{(m+n)^2 + |m-n|\,(2mn - (m+n))}{(m+n)}.$$

Proof In this graph, the Laplacian spectrum is $\mu_1 = 0$, the multiplicity of the eigen values m as $n-1$, the multiplicity of the eigen values n as $m-1$ and $\mu_{m+n} = m+n$.

The Laplacian energy

$$
\begin{aligned}
LE(K_{m,n}) &= \sum_{i=1}^{n+m}\left|\mu_i - \frac{2mn}{m+n}\right| \\
&= |0-\frac{2mn}{m+n}| + (n-1)|m-\frac{2mn}{m+n}| \\
&\quad +(m-1)|n-\frac{2mn}{m+n}| + (m+n) - \frac{2mn}{m+n} \\
&= \frac{2mn}{m+n} + \frac{m(n-1)}{m+n}|m-n| + \frac{n(m-1)}{m+n}|n-m| \\
&= \frac{(m+n)^2+|m-n|(2mn-(m+n))}{m+n}. \qquad \square
\end{aligned}
$$

Corollary 3.5 *The Laplacian energy of a star graph $K_{1,n}$ is* $\dfrac{2(n^2+1)}{n+1}$.

Proof Let m be replaced by one in Theorem 3.4. We get the following

$$LE(K_{1,n}) = \frac{(1+n)^2+|1-n|\,(2n-(1+n))}{1+n} = \frac{2(n^2+1)}{n+1}. \qquad \square$$

§4. The Laplacian Energy of Paths P_n and Cycles C_n

Definition 4.1 *A path P_n with n vertices has $V(P_n)=\{v_1,v_2,\cdots,v_n\}$ for its vertex set and $E(P_n)=\{v_1v_2,v_2v_3,\cdots,v_{n-1}v_n\}$ is its edge set. This path P_n is said to have length $n-1$.*

Definition 4.2 *A cycle C_n with n points is a graph with vertex set $V(C_n)=\{v_1,v_2,\cdots,v_n\}$ and edge set $E(C_n)=\{v_1v_2,v_2v_3,\cdots,v_{n-1}v_n,v_nv_1\}$.*

Theorem 4.3 *The Laplacian energy of the path P_n with n vertices is* $\sum\limits_{i=0}^{n-1}\left|2\left[\frac{1}{n}-cos\left(\frac{\pi i}{n}\right)\right]\right|$.

Proof The eigen values of the Laplacian matrix of P_n are $2\left[1-cos\left(\frac{\pi i}{n}\right)\right]$, $i=0,1,\cdots,n-1$. Then,

$$LE(P_n) = \sum_{i=0}^{n-1}\left|2\left[1-cos\left(\frac{\pi i}{n}\right)\right]-\frac{2(n-1)}{n}\right| = \sum_{i=0}^{n-1}\left|2\left[\frac{1}{n}-cos\left(\frac{\pi i}{n}\right)\right]\right|. \qquad \square$$

Theorem 4.4 *The Laplacian energy of the cycle C_n with n vertices is* $2\sum_{i=0}^{n-1}\left|cos\left(\frac{2\pi i}{n}\right)\right|$.

Proof The Laplacian spectrum of the cycle C_n is $2\left[1-cos\left(\frac{2\pi i}{n}\right)\right]$, $i=0,1,\cdots,(n-1)$. Then

$$LE(C_n) = \sum_{i=0}^{n-1}\left|2\left[1-cos\left(\frac{2\pi i}{n}\right)\right]-2\right| = 2\sum_{i=0}^{n-1}\left|cos\left(\frac{2\pi i}{n}\right)\right|. \qquad \square$$

§5. The Laplacian Energy of Friendship Graphs

Definition 5.1([6]) *The friendship graph $F_r (r \geq 1)$ consists of r triangles with a common vertex.*

Illustration. The Friendship graph F_4 consists of 4 triangles with a common vertex is as shown in Fig.1.

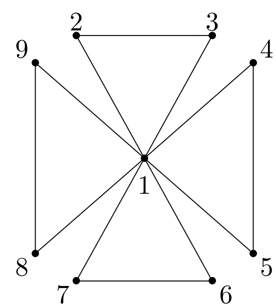

Fig.1 Friendship graph F_4

The Laplacian matrix of F_2 is

$$\begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 \\ -1 & 0 & 0 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}$$

Theorem 5.2 *The Laplacian energy of the friendship graph F_r is $\dfrac{8r^2+2r+2}{2r+1}$, where $r \geq 1$.*

Proof The friendship graph F_r has $2r+1$ vertices and $3r$ edges. Its Laplacian matrix has $2r+1$ eigen values. These eigen values are $\mu_1 = 2r+1$, the multiplicity of the eigen value 3 as r, the multiplicity of the eigen value 1 as $r-1$ and $\mu_{2r+1} = 0$.

By definition, the Laplacian energy

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|.$$

Thus,

$$\begin{aligned} LE(F_r) &= \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right| = \sum_{i=1}^{n} \left| \mu_i - \frac{6r}{2r+1} \right| \\ &= \left| 2r+1 - \frac{6r}{2r+1} \right| + r \left| 3 - \frac{6r}{2r+1} \right| + (r-1) \left| 1 - \frac{6r}{2r+1} \right| + \left| 0 - \frac{6r}{2r+1} \right| \\ &= \left| \frac{4r^2-2r+1}{2r+1} \right| + r\frac{3}{2r+1} + (r-1) \left| \frac{4r-1}{2r+1} \right| + \frac{6r}{2r+1} = \frac{8r^2+2r+2}{2r+1} \end{aligned}$$

since $4r^2+1>2r$ and $1-4r<0$. □

Corollary 5.1 *If G is the friendship graph of n vertices then* $LE(G)=\dfrac{2n^2-3n+3}{n}$.

Proof Replacing r by $\dfrac{n-1}{2}$ in Theorem 5.2, we get the result. □

Corollary 5.2 *If G is the friendship graph of m edges then* $LE(G)=\dfrac{2}{3}\left[\dfrac{4m^2+3m+9}{2m+3}\right]$.

Proof Let r be replaced by $\dfrac{m}{3}$ in Theorem 5.2, we get the result. From [3], $M=m+\dfrac{1}{2}\sum_{i=1}^{n}\left(d_i-\dfrac{2m}{n}\right)^2$. In a friendship graph $M=\dfrac{r}{2r+1}\left(4r^2-2r+7\right)$. Therefore, $2Mn=2r\left(4r^2-2r+7\right)$. Hence, using Property 1.3, we get the following

$$2\sqrt{\frac{r}{2r+1}\left(4r^2-2r+7\right)}\leq LE(G)\leq\frac{2r}{2r+1}\left(4r^2-2r+7\right).$$

□

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Super Mean Labeling of Some Classes of Graphs

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Abstract: Let G be a (p,q) graph and $f : V(G) \to \{1, 2, 3, \ldots, p+q\}$ be an injection. For each edge $e = uv$, let $f^*(e) = (f(u) + f(v))/2$ if $f(u) + f(v)$ is even and $f^*(e) = (f(u) + f(v)+1)/2$ if $f(u)+f(v)$ is odd. Then f is called a super mean labeling if $f(V)\cup\{f^*(e) : e \in E(G)\} = \{1, 2, 3, \ldots, p+q\}$. A graph that admits a super mean labeling is called a super mean graph. In this paper we prove that $S(P_n\odot K_1), S(P_2\times P_4), S(B_{n,n}), \langle B_{n,n} : P_m\rangle, C_n\odot\overline{K_2}, n \geq 3$, generalized antiprism $\mathcal{A}_n^m$ and the double triangular snake $D(T_n)$ are super mean graphs.

Key Words: Smarandachely super m-mean labeling, Smarandachely super m-mean graph, super mean labeling, super mean graph.

AMS(2010): 05C78

§1. Introduction

By a graph we mean a finite, simple and undirected one. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. The disjoint union of two graphs G_1 and G_2 is the graph $G_1\cup G_2$ with $V(G_1\cup G_2) = V(G_1)\cup V(G_2)$ and $E(G_1\cup G_2) = E(G_1)\cup E(G_2)$. The disjoint union of m copies of the graph G is denoted by mG. The corona $G_1 \odot G_2$ of the graphs G_1 and G_2 is obtained by taking one copy of G_1 (with p vertices) and p copies of G_2 and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2. Armed crown $C_n\Theta P_m$ is a graph obtained from a cycle C_n by identifying the pendent vertex of a path P_m at each vertex of the cycle. Bi-armed crown is a graph obtained from a cycle C_n by identifying the pendant vertices of two vertex disjoint paths of equal length $m-1$ at each vertex of the cycle. We denote a bi-armed crown by $C_n\Theta 2P_m$, where P_m is a path of length $m-1$. The double triangular snake $D(T_n)$ is the graph obtained from the path $v_1, v_2, v_3, \ldots, v_n$ by joining v_i and v_{i+1} with two new vertices i_i and w_i for $1 \leq i \leq n-1$. The bistar $B_{m,n}$ is a graph obtained from

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K_2 by joining m pendant edges to one end of K_2 and n pendant edges to the other end of K_2. The generalized prism graph $C_n \times P_m$ has the vertex set $V = \{v_i^j : 1 \leq i \leq n \text{ and } 1 \leq j \leq m\}$ and the edge set $E = \{v_i^j v_{i+1}^j, v_n^j v_1^j : 1 \leq i \leq n-1 \text{ and } 1 \leq j \leq m\} \cup \{v_i^j v_{i-1}^{j+1}, v_1^j v_n^{j+1} : 2 \leq i \leq n$ and $1 \leq j \leq m-1\}$. The generalized antiprism $\mathcal{A}_n^m$ is obtained by completing the generalized prism $C_n \times P_m$ by adding the edges $v_i^j v_i^{j+1}$ for $1 \leq i \leq n$ and $1 \leq j \leq m-1$. Terms and notations not defined here are used in the sense of Harary [1].

§2. Preliminary Results

Let G be a graph and $f : V(G) \to \{1, 2, 3, \cdots, |V| + |E(G)|\}$ be an injection. For each edge $e = uv$ and an integer $m \geq 2$, the induced Smarandachely edge m-labeling f_S^* is defined by

$$f_S^*(e) = \left\lceil \frac{f(u) + f(v)}{m} \right\rceil.$$

Then f is called a Smarandachely super m-mean labeling if $f(V(G)) \cup \{f^*(e) : e \in E(G)\} = \{1, 2, 3, \cdots, |V| + |E(G)|\}$. A graph that admits a Smarandachely super mean m-labeling is called Smarandachely super m-mean graph. Particularly, if $m = 2$, we know that

$$f^*(e) = \begin{cases} \frac{f(u)+f(v)}{2} & \text{if } f(u) + f(v) \text{ is even;} \\ \frac{f(u)+f(v)+1}{2} & \text{if } f(u) + f(v) \text{ is odd.} \end{cases}$$

Such a labeling f is called a super mean labeling of G if $f(V(G)) \cup (f^*(e) : e \in E(G)\} = \{1, 2, 3, \ldots, p + q\}$. A graph that admits a super mean labeling is called a super mean graph. The concept of super mean labeling was introduced in [7] and further discussed in [2-6].

We use the following results in the subsequent theorems.

Theorem 2.1([7]) *The bistar $B_{m,n}$ is a super mean graph for $m = n$ or $n + 1$.*

Theorem 2.2([2]) *The graph $\langle B_{n,n} : w \rangle$, obtained by the subdivision of the central edge of $B_{n,n}$ with a vertex w, is a super mean graph.*

Theorem 2.3([2]) *The bi-armed crown $C_n \Theta 2P_m$ is a super mean graph for odd $n \geq 3$ and $m \geq 2$.*

Theorem 2.4([7]) *Let $G_1 = (p_1, q_1)$ and $G_2 = (p_2, q_2)$ be two super mean graphs with super mean labeling f and g respectively. Let $f(u) = p_1 + q_1$ and $g(v) = 1$. Then the graph $(G_1)_f * (G_2)_g$ obtained from G_1 and G_2 by identifying the vertices u and v is also a super mean graph.*

§3. Super Mean Graphs

If G is a graph, then $S(G)$ is a graph obtained by subdividing each edge of G by a vertex.

Theorem 3.1 *The graph $S(P_n \odot K_1)$ is a super mean graph.*

Proof Let $V(P_n \odot K_1) = \{u_i, v_i : 1 \le i \le n\}$. Let $x_i (1 \le i \le n)$ be the vertex which divides the edge $u_i v_i (1 \le i \le n)$ and $y_i (1 \le i \le n-1)$ be the vertex which divides the edge $u_i u_{i+1} (1 \le i \le n-1)$. Then $V(S(P_n \odot K_1)) = \{u_i, v_i, x_i, y_j : 1 \le i \le n, 1 \le j \le n-1\}$.

Define $f : V(S(P_n \odot K_1)) \to \{1, 2, 3, \ldots, p + q = 8n - 3\}$ by

$$f(v_1) = 1; f(v_2) = 14; f(v_{2+i}) = 14 + 8i \text{ for } 1 \le i \le n-4;$$
$$f(v_{n-1}) = 8n - 11; f(v_n) = 8n - 10; f(x_1) = 3;$$
$$f(x_{1+i}) = 3 + 8i \text{ for } 1 \le i \le n-2; f(x_n) = 8n - 7;$$
$$f(u_1) = 5; f(u_2) = 9; f(u_{2+i}) = 9 + 8i \text{ for } 1 \le i \le n-3;$$
$$f(u_n) = 8n - 5; f(y_i) = 8i - 1 \text{ for } 1 \le i \le n-2; f(y_{n-1}) = 8n - 3.$$

It can be verified that f is a super mean labeling of $S(P_n \odot K_1)$. Hence $S(P_n \odot K_1)$ is a super mean graph. □

Example 3.2 The super mean labeling of $S(P_5 \odot K_1)$ is given in Fig.1.

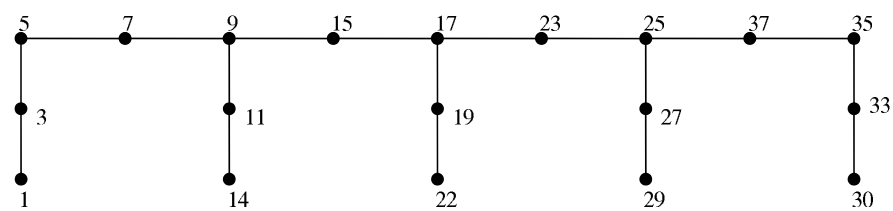

Fig.1

Theorem 3.2 *The graph $S(P_2 \times P_n)$ is a super mean graph.*

Proof Let $V(P_2 \times P_n) = \{u_i, v_i : 1 \le i \le n\}$. Let $u_i^1, v_i^1 (1 \le i \le n-1)$ be the vertices which divide the edges $u_i u_{i+1}, v_i v_{i+1} (1 \le i \le n-1)$ respectively. Let $w_i (1 \le i \le n)$ be the vertex which divides the edge $u_i v_i$. That is $V(S(P_2 \times P_n)) = \{u_i, v_i, w_i : 1 \le i \le n\} \cup \{u_i^1, v_i^1 : 1 \le i \le n-1\}$.

Define $f : V(S(P_2 \times P_n)) \to \{1, 2, 3, \ldots, p + q = 11n - 6\}$ by

$$f(u_1) = 1; f(u_2) = 9; f(u_3) = 27;$$
$$f(u_i) = f(u_{i-1}) + 5 \text{ for } 4 \le i \le n \text{ and } i \text{ is even}$$
$$f(u_i) = f(u_{i-1}) + 17 \text{ for } 4 \le i \le n \text{ and } i \text{ is odd}$$
$$f(v_1) = 7; f(v_2) = 16;$$
$$f(v_i) = f(v_{i-1}) + 5 \text{ for } 3 \le i \le n \text{ and } i \text{ is odd}$$
$$f(v_i) = f(v_{i-1}) + 17 \text{ for } 3 \le i \le n \text{ and } i \text{ is even}$$
$$f(w_1) = 3; f(w_2) = 12;$$
$$f(w_{2+i}) = 12 + 11i \text{ for } 1 \le i \le n-2;$$
$$f(u_1^1) = 6; f(u_2^1) = 24;$$

$f(u_i^1) = f(u_{i-1}^1) + 6$ for $3 \le i \le n-1$ and i isodd
$f(u_i^1) = f(u_{i-1}^1) + 16$ for $3 \le i \le n-1$ and i iseven
$f(v_1^1) = 13; f(v_i^1) = f(v_{i-1}^1) + 6$ for $2 \le i \le n-1$ and i iseven
$f(v_i^1) = f(v_{i-1}^1) + 16$ for $2 \le i \le n-1$ and i isodd.

It is easy to check that f is a super mean labeling of $S(P_2 \times P_n)$. Hence $S(P_2 \times P_n)$ is a super mean graph. □

Example 3.4 The super mean labeling of $S(P_2 \times P_6)$ is given in Fig.2.

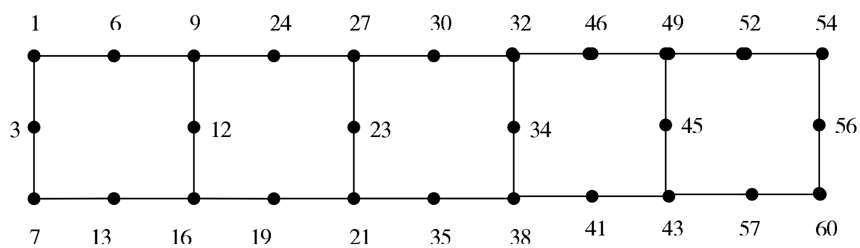

Fig.2

Theorem 3.5 *The graph $S(B_{n,n})$ is a super mean graph.*

Proof Let $V(B_{n,n}) = \{u, u_i, v, v_i : 1 \le i \le n\}$ and $E(B_{n,n}) = \{uu_i, vv_i, uv : 1 \le i \le n\}$. Let $w, x_i, y_i, (1 \le i \le n)$ be the vertices which divide the edges $uv, uu_i, vv_i (1 \le i \le n)$ respectively. Then $V(S(B_{n,n})) = \{u, u_i, v, v_i, x_i, y_i, w : 1 \le i \le n\}$ and $E(S(B_{n,n})) = \{ux_i, x_iu_i, uw, wv, vy_i, y_iv_i : 1 \le i \le n\}$.

Define $f : V(S(B_{n,n})) \to \{1, 2, 3, \ldots, p+q = 8n+5\}$ by

$f(u) = 1; f(x_i) = 8i-5$ for $1 \le i \le n; f(u_i) = 8i-3$ for $1 \le i \le n; f(w) = 8n+3$; $f(v) = 8n+5; f(y_i) = 8i-1$ for $1 \le i \le n; f(v_i) = 8i+1$ for $1 \le i \le n$. It can be verified that f is a super mean labeling of $S(B_{n,n})$. Hence $S(B_{n,n})$ is a super mean graph. □

Example 3.6 The super mean labeling of $S(B_{n,n})$ is given in Fig.3.

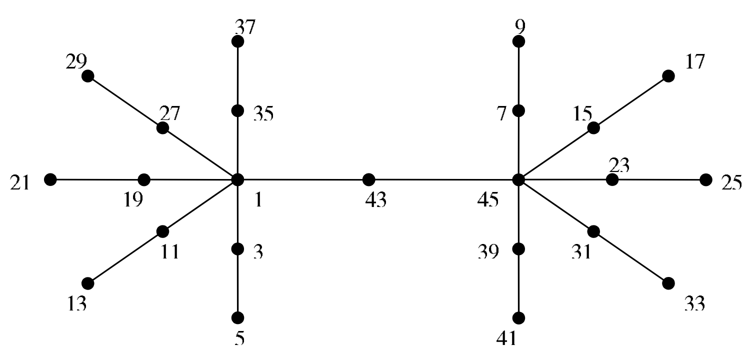

Fig.3

Next we prove that the graph $\langle B_{n,n} : P_m \rangle$ is a super mean graph. $\langle B_{m,n} : P_k \rangle$ is a graph obtained by joining the central vertices of the stars $K_{1,m}$ and $K_{1,n}$ by a path P_k of length $k-1$.

Theorem 3.7 *The graph $\langle B_{n,n} : P_m \rangle$ is a super mean graph for all $n \geq 1$ and $m > 1$.*

Proof Let $V(\langle B_{n,n} : P_m \rangle) = \{u_i, v_i, u, v, w_j : 1 \leq i \leq n, 1 \leq j \leq m \text{ with } u = w_1, v = w_m\}$ and $E(\langle B_{n,n} : P_m \rangle) = \{uu_i, vv_i, w_j w_{j+1} : 1 \leq i \leq n, 1 \leq j \leq m-1\}$.

Case 1 n is even.

Subcase 1 m is odd.

By Theorem 2.2, $\langle B_{n,n} : P_3 \rangle$ is a super mean graph. For $m > 3$, define $f : V(\langle B_{n,n} : P_m \rangle) \rightarrow \{1, 2, 3, \ldots, p+q = 4n+2m-1\}$ by

$$f(u) = 1; f(u_i) = 4i-1 \text{ for } 1 \leq i \leq n \text{ and for } i \neq \frac{n}{2}+1;$$
$$f(u_{\frac{n}{2}+1}) = 2n+2; f(v_i) = 4i+1 \text{ for } 1 \leq i \leq n; f(v) = 4n+3;$$
$$f(w_2) = 4n+4; f(w_3) = 4n+9;$$
$$f(w_{3+i}) = 4n+9+4i \text{ for } 1 \leq i \leq \frac{m-5}{2}; f(w_{\frac{m+3}{2}}) = 4n+2m-4$$
$$f(w_{\frac{m+3}{2}+i}) = 4n+2m-4-4i \text{ for } 1 \leq i \leq \frac{m-5}{2}.$$

It can be verified that f is a super mean labeling of $\langle B_{n,n} : P_m \rangle$.

Subcase 2 m is even.

By Theorem 2.1, $\langle B_{n,n} : P_2 \rangle$ is a super mean graph. For $m > 2$, define $f : V(\langle B_{n,n} : P_m \rangle) \rightarrow \{1, 2, 3, \ldots, p+q = 4n+2m-1\}$ by

$$f(u) = 1; f(u_i) = 4i-1 \text{ for } 1 \leq i \leq n \text{ and for } i \neq \frac{n}{2}+1;$$
$$f(u_{\frac{n}{2}+1}) = 2n+2; f(v_i) = 4i+1 \text{ for } 1 \leq i \leq n; f(v) = 4n+3;$$
$$f(w_2) = 4n+4; f(w_{2+i}) = 4n+4+2i \text{ for } 1 \leq i \leq \frac{m-4}{2};$$
$$f(w_{\frac{m+2}{2}}) = 4n+m+3;$$
$$f(w_{\frac{m+2}{2}+i}) = 4n+m+3+2i \text{ for } 1 \leq i \leq \frac{m-4}{2}.$$

It can be verified that f is a super mean labeling of $\langle B_{n,n} : P_m \rangle$.

Case 2 n is odd.

Subcase 1 m is odd.

By Theorem 2.1, $\langle B_{n,n} : P_2 \rangle$ is a super mean graph. For $m > 2$, define $f : V(\langle B_{n,n} : P_m \rangle) \rightarrow$

$\{1, 2, 3, \ldots, p + q = 4n + 2m - 1\}$ by

$$f(u) = 1; f(v) = 4n + 3; f(u_i) = 4i - 1 \text{ for } 1 \leq i \leq n;$$
$$f(v_i) = 4i + 1 \text{ for } 1 \leq i \leq n \text{ and for } i \neq \frac{n+1}{2};$$
$$f(v_{\frac{n+1}{2}}) = 2n + 2; f(w_2) = 4n + 4;$$
$$f(w_{2+i}) = 4n + 4 + 2i \text{ for } 1 \leq i \leq \frac{m-4}{2}; f(w_{\frac{m+2}{2}}) = 4n + m + 3;$$
$$f(w_{\frac{m+2}{2}+i}) = 4n + m + 3 + 2i \text{ for } 1 \leq i \leq \frac{m-4}{2}.$$

It can be verified that f is a super mean labeling of $\langle B_{n,n} : P_m \rangle$.

Subcase 2 m is even.

By Theorem 2.2, $\langle B_{n,n} : P_3 \rangle$ is a super mean graph. For $m > 3$, define $f : V(\langle B_{n,n} : P_m \rangle) \rightarrow \{1, 2, 3, \ldots, p + q = 4n + 2m - 1\}$ by

$$f(u) = 1; f(v) = 4n + 3; f(u_i) = 4i - 1 \text{ for } 1 \leq i \leq n;$$
$$f(v_i) = 4i + 1 \text{ for } 1 \leq i \leq n \text{ and for } i \neq \frac{n+1}{2}; f(v_{\frac{n+1}{2}}) = 2n + 2;$$
$$f(w_2) = 4n + 4; f(w_3) = 4n + 9;$$
$$f(w_{3+i}) = 4n + 9 + 4i \text{ for } 1 \leq i \leq \frac{m-5}{2}; f(w_{\frac{m+3}{2}}) = 4n + 2m - 4;$$
$$f(w_{\frac{m+3}{2}+i}) = 4n + 2m - 4 - 2i \text{ for } 1 \leq i \leq \frac{m-5}{2}.$$

It can be verified that f is a super mean labeling of $\langle B_{n,n} : P_m \rangle$. Hence $\langle B_{n,n} : P_m \rangle$ is a super mean graph for all $n \geq 1$ and $m > 1$. □

Example 3.8 The super mean labeling of $\langle B_{4,4} : P_5 \rangle$ is given in Fig.4.

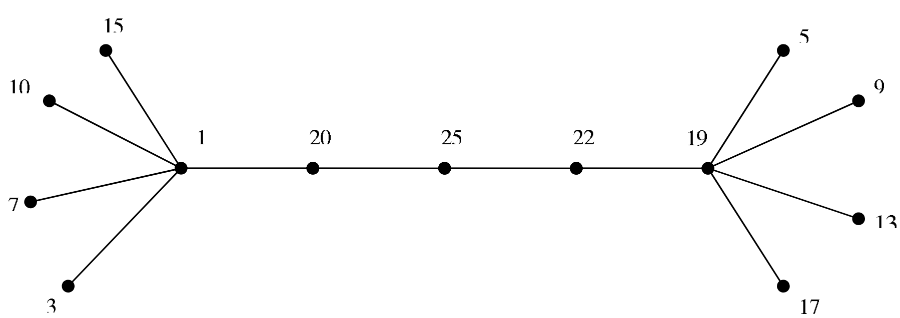

Fig.4

Theorem 3.9 *The corona graph $C_n \odot \overline{K_2}$ is a super mean graph for all $n \geq 3$.*

Proof Let $V(C_n) = \{u_1, u_2, \ldots, u_n\}$ and $V(C_n \odot \overline{K_2}) = \{u_i, v_i, w_i : 1 \leq i \leq n\}$. Then $E(C_n \odot \overline{K_2}) = \{u_iu_{i+1}, u_nu_1, u_jv_j, u_jw_j : 1 \leq i \leq n-1 \text{ and } 1 \leq j \leq n\}$.

Case 1 n is odd.

The proof follows from Theorem 2.3 by taking $m = 2$.

Case 2 n is even.

Take $n = 2k$ for some k. Define $f : V(C_n \odot \overline{K_2}) \to \{1, 2, 3, \ldots, p + q = 6n\}$ by

$f(u_i) = 6i - 3$ for $1 \leq i \leq k - 1$; $f(u_k) = 6k - 2$;
$f(u_{k+i}) = 6k - 2 + 6i$ for $1 \leq i \leq k - 2$; $f(u_{2k-1}) = 12k - 2$;
$f(u_{2k}) = 12k - 9$; $f(v_i) = 6i - 5$ for $1 \leq i \leq k - 1$; $f(v_k) = 6k - 6$;
$f(v_{k+1}) = 6k + 2$; $f(v_{k+1+i}) = 6k + 2 + 6i$ for $1 \leq i \leq k - 3$; $f(v_{2k-1}) = 12k$;
$f(v_{2k}) = 12k - 6$; $f(w_i) = 6i - 1$ for $1 \leq i \leq k - 1$; $f(w_k) = 6k$;
$f(w_{k+i}) = 6k + 6i$ for $1 \leq i \leq k - 2$; $f(w_{2k-1}) = 12k - 4$;
$f(w_{2k}) = 12k - 11$.

It can be verified that $f(V) \cup (f^*(e) : e \in E\} = \{1, 2, 3, \ldots, 6n\}$. Hence $C_n \odot \overline{K_2}$ is a super mean graph. □

Example 3.10 The super mean labeling of $C_8 \odot \overline{K_2}$ is given in Fig.5.

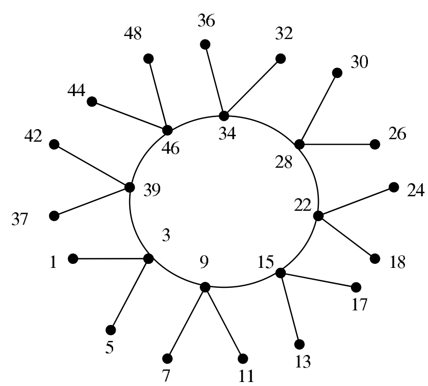

Fig.5

Theorem 3.11 *The double triangular snake $D(T_n)$ is a super mean graph.*

Proof We prove this result by induction on n. A super mean labeling of $G_1 = D(T_2)$ is given in Fig.6.

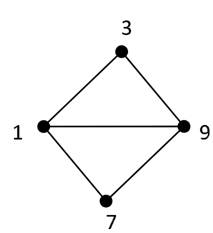

Fig.6

Therefore the result is true for $n = 2$. Let f be the super mean labeling of G_1 as in the above figure. Now $D(T_3) = (G_1)_f * (G_1)_f$, by Theorem 2.4, $D(T_3)$ is a super mean graph. Therefore the result is true for $n = 3$. Assume that $D(T_{n-1})$ is a super mean graph with the super mean labeling g. Now by Theorem 2.4, $(D(T_{n-1}))_g * (G_1)_f = D(T_n)$ is a super mean graph. Therefore the result is true for n. Hence by induction principle the result is true for all n. Thus $D(T_n)$ is a super mean graph. □

Example 3.12 The super mean labeling of $D(T_6)$ is given in Fig.7.

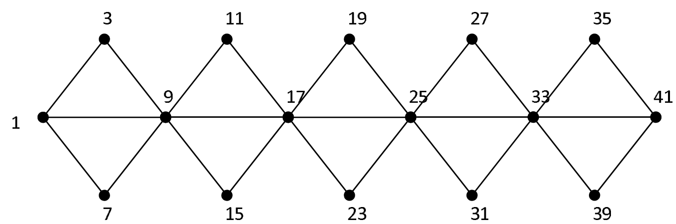

Fig.7

Theorem 3.13 *The generalized antiprism $\mathcal{A}_n^m$ is a super mean graph for all $m \geq 2, n \geq 3$ except for $n = 4$.*

Proof Let $V(\mathcal{A}_n^m) = \{v_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(\mathcal{A}_n^m) = \{v_i^j v_{i+1}^j, v_n^j v_1^j : 1 \leq i \leq n-1, 1 \leq j \leq m\} \cup \{v_i^j v_{i-1}^{j+1}, v_1^j v_n^{j+1} : 2 \leq i \leq n, 1 \leq j \leq m-1\} \cup \{v_i^j v_i^{j+1} : 1 \leq i \leq n$ and $1 \leq j \leq m-1\}$.

Case 1 n is odd.

Define $f : V(\mathcal{A}_n^m) \to \{1, 2, 3, \ldots, p+q = 4mn - 2n\}$ by

$$f(v_i^j) = 4(j-1)n + 2i - 1 \text{ for } 1 \leq i \leq \frac{n+1}{2} \text{ and } 1 \leq j \leq m;$$
$$f(v_{\frac{n+3}{2}}^j) = 4(j-1)n + n + 3 \text{ for } 1 \leq j \leq m;$$
$$f(v_{\frac{n+3}{2}+i}^j) = 4(j-1)n + n + 3 + 2i \text{ for } 1 \leq i \leq \frac{n-3}{2} \text{ and } 1 \leq j \leq m.$$

Then f is a super mean labeling of $\mathcal{A}_n^m$. Hence $\mathcal{A}_n^m$ is a super mean graph.

Case 2 n is even and $n \neq 4$.

Define $f : V(\mathcal{A}_n^m) \to \{1, 2, 3, \ldots, p+q = 4mn - 2n\}$ by

$$f(v_1^j) = 4(j-1)n + 1 \text{ for } 1 \leq j \leq m; f(v_2^j) = 4(j-1)n + 3 \text{ for } 1 \leq j \leq m;$$
$$f(v_3^j) = 4(j-1)n + 7 \text{ for } 1 \leq j \leq m; f(v_4^j) = 4(j-1)n + 12 \text{ for } 1 \leq j \leq m;$$
$$f(v_{4+i}^j) = 4(j-1)n + 12 + 4i \text{ for } 1 \leq j \leq m \text{ and } 1 \leq i \leq \frac{n-6}{2};$$
$$f(v_{\frac{n+2}{2}+i}^j) = 4(j-1)n + 2n + 1 - 4i \text{ for } 1 \leq j \leq m \text{ and } 1 \leq i \leq \frac{n-6}{2};$$
$$f(v_{n-1}^j) = 4(j-1)n + 9 \text{ for } 1 \leq j \leq m; f(v_n^j) = 4(j-1)n + 6 \text{ for } 1 \leq j \leq m.$$

Then f is a super mean labeling of $\mathcal{A}_n^m$. Hence $\mathcal{A}_n^m$ is a super mean graph. □

Example 3.14 The super mean labeling of $\mathcal{A}_6^3$ is given in Fig.8.

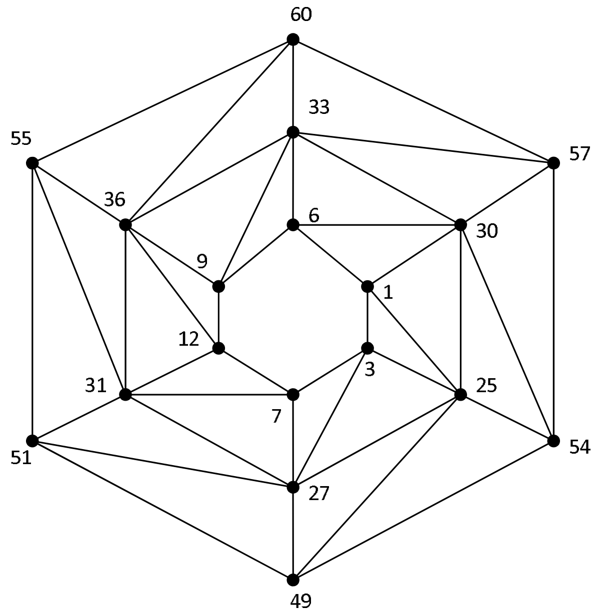

Fig.8

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The t-Pebbling Number of Jahangir Graph

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Abstract: Given a configuration of pebbles on the vertices of a connected graph G, a pebbling move (or pebbling step) is defined as the removal of two pebbles from a vertex and placing one pebble on an adjacent vertex. The t-pebbling number, $f_t(G)$ of a graph G is the least number m such that, however m pebbles are placed on the vertices of G, we can move t pebbles to any vertex by a sequence of pebbling moves. In this paper, we determine $f_t(G)$ for Jahangir graph $J_{2,m}$.

Key Words: Smarandachely d-pebbling move, Smarandachely d-pebbling number, pebbling move, t-pebbling number, Jahangir graph.

AMS(2010): 05C78

§1. Introduction

One recent development in graph theory, suggested by Lagarias and Saks, called pebbling, has been the subject of much research. It was first introduced into the literature by Chung [1], and has been developed by many others including Hulbert, who published a survey of pebbling results in [2]. There have been many developments since Hulbert's survey appeared.

Given a graph G, distribute k pebbles (indistinguishable markers) on its vertices in some configuration C. Specifically, a configuration on a graph G is a function from $V(G)$ to $N \cup \{0\}$ representing an arrangement of pebbles on G. For our purposes, we will always assume that G is connected. A *Smarandachely d-pebbling move* (Smarandachely d-pebbling step) is defined as the removal of two pebbles from some vertex and the replacement of one of these pebbles on such a vertex with distance d to the initial vertex with pebbles and the Smarandachely (t, d)-pebbling number $f_t^d(G)$, is defined to be the minimum number of pebbles such that regardless of their initial configuration, it is possible to move to any root vertex v, t pebbles by a sequence

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of Smarandachely d-pebbling moves. Particularly, if $d = 1$, such a Smarandachely 1-pebbling move is called a pebbling move (or pebbling step)and The Smarandache $(t, 1)$-pebbling number $f_t^d(G)$ is abbreviated to $f_t(G)$, i.e., it is possible to move to any root vertex v, t pebbles by a sequence of pebbling moves. Implicit in this definition is the fact that if after moving to vertex v one desires to move to another root vertex, the pebbles reset to their original configuration. There are certain results regarding the t-pebbling graphs that are investigated in [3-6,9].

Definition 1.1 *Jahangir graph $J_{n,m}$ for $m \geq 3$ is a graph on $nm + 1$ vertices, that is, a graph consisting of a cycle C_{nm} with one additional vertex which is adjacent to C_{nm}.*

Example 1.2 Fig.1 shows Jahangir graph $J_{2,8}$. The graph $J_{2,8}$ appears on Jahangir's tomb in his mausoleum. It lies in 5 kilometer north- west of Lahore, Pakistan, across the River Ravi.

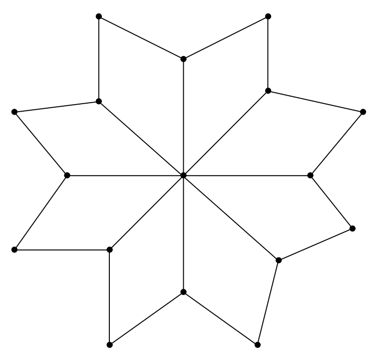

Fig.1 $J_{2,8}$

Remark 1.3 *Let v_{2m+1} be the label of the center vertex and $v_1, v_2, \cdots, v_{2m}$ be the label of the vertices that are incident clockwise on cycle C_{2m} so that $deg(v_1) = 3$.*

In Section 2, we determine the t-pebbling number for Jahangir graph $J_{2,m}$. For that we use the following theorems.

Theorem 1.4([7]) *For the Jahangir graph $J_{2,3}, f(J_{2,3}) = 8$.*

Theorem 1.5([7]) *For the Jahangir graph $J_{2,4}, f(J_{2,4}) = 16$.*

Theorem 1.6([7]) *For the Jahangir graph $J_{2,5}, f(J_{2,5}) = 18$.*

Theorem 1.7([7]) *For the Jahangir graph $J_{2,6}, f(J_{2,6}) = 21$.*

Theorem 1.8([7]) *For the Jahangir graph $J_{2,7}, f(J_{2,7}) = 23$.*

Theorem 1.9([8]) *For the Jahangir graph $J_{2,m}(m \geq 8), f(J_{2,m}) = 2m + 10$.*

We now proceed to find the t-pebbling number for $J_{2,m}$.

§2. The t-Pebbling Number for Jahangir Graph $J_{2,m}, m \geq 3$

Theorem 2.1 *For the Jahangir graph* $J_{2,3}$, $f_t(J_{2,3}) = 8t$.

Proof Consider the Jahangir graph $J_{2,3}$. We prove this theorem by induction on t. By Theorem 1.4, the result is true for $t = 1$. For $t > 1$, $J_{2,3}$ contains at least 16 pebbles. Using at most 8 pebbles, we can put a pebble on any desired vertex, say $v_i(1 \leq i \leq 7)$, by Theorem 1.4. Then, the remaining number of pebbles on the vertices of $J_{2,3}$ is at least $8t - 8$. By induction we can put $t - 1$ additional pebbles on the desired vertex $v_i(1 \leq i \leq 7)$. So, the result is true for all t. Thus, $f_t(J_{2,3}) \leq 8t$.

Now, consider the following configuration C such that $C(v_4) = 8t-1$, and $C(x) = 0$, where $x \in V \setminus \{v_4\}$, then we cannot move t pebbles to the vertex v_1. Thus, $f_t(J_{2,3}) \geq 8t$. Therefore, $f_t(J_{2,3}) = 8t$. □

Theorem 2.2 *For the Jahangir graph* $J_{2,4}$, $f_t(J_{2,4}) = 16t$.

Proof Consider the Jahangir graph $J_{2,4}$. We prove this theorem by induction on t. By Theorem 1.5, the result is true for $t = 1$. For $t > 1$, $J_{2,4}$ contains at least 32 pebbles. By Theorem 1.5, using at most 16 pebbles, we can put a pebble on any desired vertex, say $v_i(1 \leq i \leq 9)$. Then, the remaining number of pebbles on the vertices of $J_{2,4}$ is at least $16t - 16$. By induction, we can put $t - 1$ additional pebbles on the desired vertex $v_i(1 \leq i \leq 9)$. So, the result is true for all t. Thus, $f_t(J_{2,4}) \leq 16t$.

Now, consider the following configuration C such that $C(v_6) = 16t - 1$, and $C(x) = 0$, where $x \in V \setminus \{v_6\}$, then we cannot move t pebbles to the vertex v_2. Thus, $f_t(J_{2,4}) \geq 16t$. Therefore, $f_t(J_{2,4}) = 16t$. □

Theorem 2.3 *For the Jahangir graph* $J_{2,5}$, $f_t(J_{2,5}) = 16t + 2$.

Proof Consider the Jahangir graph $J_{2,5}$. We prove this theorem by induction on t. By Theorem 1.6, the result is true for $t = 1$. For $t > 1$, $J_{2,5}$ contains at least 34 pebbles. Using at most 16 pebbles, we can put a pebble on any desired vertex, say $v_i(1 \leq i \leq 11)$. Then, the remaining number of pebbles on the vertices of the graph $J_{2,5}$ is at least $16t-14$. By induction, we can put $t - 1$ additional pebbles on the desired vertex $v_i(1 \leq i \leq 11)$. So, the result is true for all t. Thus, $f_t(J_{2,5}) \leq 16t + 2$.

Now, consider the following distribution C such that $C(v_6) = 16t-1$, $C(v_8) = 1$, $C(v_{10}) = 1$ and $C(x) = 0$, where $x \in V \setminus \{v_6, v_8, v_{10}\}$. Then we cannot move t pebbles to the vertex v_2. Thus, $f_t(J_{2,5}) \geq 16t + 2$. Therefore, $f_t(J_{2,5}) = 16t + 2$. □

Theorem 2.4 *For the Jahangir graph* $J_{2,m}(m \geq 6)$, $f_t(J_{2,m}) = 16(t - 1) + f(J_{2,m})$.

Proof Consider the Jahangir graph $J_{2,m}$, where $m > 5$. We prove this theorem by induction on t. By Theorems $1.7 - 1.9$, the result is true for $t = 1$. For $t > 1$, $J_{2,m}$ contains at least $16 + f(J_{2,m}) = 16 + \begin{cases} 2m+9 & m = 6,7 \\ 2m+10 & m \geq 8. \end{cases}$ pebbles. Using at most 16 pebbles, we can put a

pebble on any desired vertex, say $v_i(1 \leq i \leq 2m+1)$. Then, the remaining number of pebbles on the vertices of the graph $J_{2,m}$ is at least $16t + f(J_{2,m}) - 32$. By induction, we can put $t-1$ additional pebbles on the desired vertex $v_i(1 \leq i \leq 2m+1)$. So, the result is true for all t. Thus, $f_t(J_{2,m}) \leq 16(t-1) + f(J_{2,m})$.

Now, consider the following distributions on the vertices of $J_{2,m}$.

For $m = 6$, consider the following distribution C such that $C(v_6) = 16(t-1)+15$, $C(v_{10}) = 3$, $C(v_8) = 1$, $C(v_{12}) = 1$ and $C(x) = 0$, where $x \in V \setminus \{v_6, v_8, v_{10}, v_{12}\}$.

For $m = 7$, consider the following distribution C such that $C(v_6) = 16(t-1)+15$, $C(v_{10}) = 3$, $C(v_8) = C(v_{12}) = C(v_{13}) = C(v_{14}) = 1$ and $C(x) = 0$, where $x \in V \setminus \{v_6, v_8, v_{10}, v_{12}, v_{13}, v_{14}\}$.

For $m \geq 8$, if m is even, consider the following distribution C_1 such that $C_1(v_{m+2}) = 16(t-1)+15$, $C_1(v_{m-2}) = 3$, $C_1(v_{m+6}) = 3$, $C_1(x) = 1$, where $x \in \{N[v_2], N[v_{m+2}], N[v_{m-2}], N[v_{m+6}]\}$ and $C_1(y) = 0$, where $y \in \{N[v_2], N(v_{m+2}), N(v_{m-2}), N(v_{m+6})\}$.

If m is odd, then consider the following configuration C_2 such that $C_2(v_{m+1}) = 16(t-1)+15$, $C_2(v_{m-3}) = 3$, $C_2(v_{m+5}) = 3$, $C_2(x) = 1$, where $x \in \{N[v_2], N[v_{m+1}], N[v_{m-3}], N[v_{m+5}]\}$ and $C_2(y) = 0$, where $y \in \{N[v_2], N(v_{m+1}), N(v_{m-3}), N(v_{m+5})\}$. Then, we cannot move t pebbles to the vertex v_2 of $J_{2,m}$ for all $m \geq 6$. Thus, $f_t(J_{2,m}) \geq 16(t-1) + f(J_{2,m})$. Therefore, $f_t(J_{2,m}) = 16(t-1) + f(J_{2,m})$. $\square$

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3-Product Cordial Labeling of Some Graphs

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Abstract: A mapping $f : V(G) \rightarrow \{0, 1, 2\}$ is called a 3-product cordial labeling if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for any $i, j \in \{0, 1, 2\}$, where $v_f(i)$ denotes the number of vertices labeled with $i, e_f(i)$ denotes the number of edges xy with $f(x)f(y) \equiv i$ $(mod\ 3)$. A graph with a 3-product cordial labeling is called a 3-product cordial graph. In this paper, we establish that the duplicating arbitrary vertex in cycle C_n, duplicating arbitrarily edge in cycle C_n, duplicating arbitrary vertex in wheel W_n, Ladder L_n, Triangular Ladder TL_n and the graph $\left\langle W_n^{(1)} : W_n^{(2)} : \cdots : W_n^{(k)} \right\rangle$ are 3-product cordial.

Key Words: Cordial labeling, Smarandachely p-product cordial labeling, 3-product cordial labeling, 3-product cordial graph.

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§1. Introduction

All graphs considered here are simple, finite, connected and undirected. We follow the basic notations and terminologies of graph theory as in [2]. The symbols $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph G. Let $G(p, q)$ be a graph with $p = |V(G)|$ vertices and $q = |E(G)|$ edges. A vertex labeling of a graph is a function from the vertex set of the graph to the natural numbers. There are several types of labeling. A detailed survey of graph labeling can be found in [5].

In 1987, Cahit introduced the idea of cordial labelings, a generalization of both graceful and harmonious labelings in [1]. Let f be a function from the vertices of G to $\{0, 1\}$ and for each edge xy assign the label $|f(x) - f(y)|$. f is called a cordial labeling of G if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1 and the number of edges labeled 0 and the number of edges labeled 1 differ at most by 1. M. Sundaram et al., introduced the concept of product cordial labeling of a graph in [6]. A product cordial labeling of a graph G with the vertex set V is a function f from V to $\{0, 1\}$ such that if each edge uv is

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assigned the label $f(u)f(v)$, the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1 and the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1.

P.Jeyanthi and A.Maheswari introduced the concept of 3-product cordial labeling [3] and further studied in [4]. They proved that the graphs like path $P_n, K_{1,n}$, $\langle B_{n,n} : w\rangle$, cycle C_n if $n \equiv 1, 2(mod\ 3)$, $C_n \cup P_n$, $Cm \circ \overline{K_n}$ if $m \geq 3$ and $n \geq 1$, $P_m \circ \overline{K_n}$ if $m, n \geq 1$, W_n if $n \equiv 1(mod\ 3)$ and the middle graph of P_n, the splitting graph of P_n, the total graph of $P_n, P_n[P_2], P_n^2, K_{2,n}$ and the vertex switching of C_n are 3-product cordial graphs. Also they proved that the complete graph K_n is not 3-product cordial if $n \geq 3$. In addition, they proved that if $G(p, q)$ is a 3-product cordial graph, then (i) if $p \equiv 1(mod\ 3)$, then $q \leq \dfrac{p^2 - 2p + 7}{3}$; (ii) if $p \equiv 2(mod\ 3)$ then $q \leq \dfrac{p^2 - p + 4}{3}$; (iii) if $p \equiv 0(mod\ 3)$ then $q \leq \dfrac{p^2 - 3p + 6}{3}$ and if G_1 is a 3-product cordial graph with $3m$ vertices and $3n$ edges and G_2 is any 3-product cordial graph then $G_1 \cup G_2$ is also 3-product cordial graph.

In this paper, we establish that the duplicating arbitrary vertex and duplicating arbitrary edge in cycle C_n, duplicating arbitrary rim vertex in wheel W_n, ladder L_n, triangular ladder TL_n and $\left\langle W_n^{(1)} : W_n^{(2)} : \cdots : W_n^{(k)} \right\rangle$ are 3-product cordial.

We use the following definitions in the subsequent section.

Definition 1.1 *Let G be a graph and let v be a vertex of G. The duplicate graph $D(G, v')$ of $G$is the graph whose vertex set is $V(G) \cup \{v'\}$ and edge set is $E(G) \cup (v'x|x$ is the vertex adjacent to v in $G\}$.*

Definition 1.2 *Let G be a graph and let $e = uv$ be an edge of G. The duplicate graph $D(G, e' = u'v')$ of G is the graph whose vertex set is $V(G) \cup \{u', v'\}$ and edge set is $E(G) \cup \{u'x, v'y|x$ and y are the vertices adjacent with u and v in G respectively $\}$.*

Definition 1.3 *Consider k copies of wheels namely $W_n^{(1)}, W_n^{(2)}, \ldots, W_n^{(k)}$. Then, the graph $G = \left\langle W_n^{(1)} : W_n^{(2)} : \cdots : W_n^{(k)} \right\rangle$ is obtained by joining apex vertex of each $W_n^{(p)}$ and apex of $W_n^{(p-1)}$ to a new vertex x_{p-1} for $2 \leq p \leq k$.*

Definition 1.4 *The ladder graph L_n is defined as the cartesian product of two path graphs.*

Definition 1.5 *A triangular ladder $TL_n, n \geq 2$ is a graph obtained by completing the ladder TL_n by edges u_iv_{i+1} for $1 \leq i \leq n - 1$.*

Definition 1.6 *Let p be an integer with $p > 1$. A mapping $f : V(G) \to \{0, 1, 2, \cdots, p\}$ is called a Smarandachely p-product cordial labeling if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for any $i, j \in \{0, 1, 2, \cdots, p-1\}$, where $v_f(i)$ denotes the number of vertices labeled with i, $e_f(i)$ denotes the number of edges xy with $f(x)f(y) \equiv i(mod\ p)$. Particularly, if $p = 3$,such a Smarandachely 3-product cordial labeling is called 3-product cordial labeling. A graph with 3-product cordial labeling is called a 3-product cordial graph.*

For any real number n, $\lceil n \rceil$ denotes the smallest integer $\geq n$ and $\lfloor n \rfloor$ denotes the greatest integer $\leq n$.

§2. Main Results

Theorem 2.1 *A duplicate graph $D(C_n, e')$ of a cycle C_n is a 3-product cordial graph.*

Proof Let $v_1, v_2, \ldots, v_n$ be the vertices of the cycle C_n and let $e = v_1v_2$ and the duplicated edge is $e' = v_1'v_2'$.

Define a vertex labeling $f : V(G) \to \{0, 1, 2\}$ by considering the following three cases.

Case 1 $n \equiv 0(mod\ 3), n > 6$.

Define $f(v_i) = \begin{cases} 2 & \text{if } i \equiv 0, 1(mod\ 4) \\ 1 & \text{if } i \equiv 2, 3(mod\ 4) \end{cases}$ for $1 \le i \le n - \left\lceil \dfrac{n+2}{3} \right\rceil - 1$,

$f(v_i) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$ for $i = n - \left\lceil \dfrac{n+2}{3} \right\rceil$ and $f(v_i) = 0$ if $n - \left\lceil \dfrac{n+2}{3} \right\rceil < i \le n-1$;

$f(v_n) = 2,\ f(v_1') = \begin{cases} 1 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd} \end{cases}$; $f(v_2') = 1$.

In view of the above labeling pattern we have $v_f(0)+1 = v_f(1) = v_f(2) = \left\lceil \dfrac{n+2}{3} \right\rceil$, $e_f(0) = e_f(1) = e_f(2) = \left\lceil \dfrac{n+2}{3} \right\rceil$.

Case 2 $n \equiv 1(mod\ 3), n > 7$.

Define $f(v_i) = \begin{cases} 2 & \text{if } i \equiv 1, 2(mod\ 4) \\ 1 & \text{if } i \equiv 0, 3(mod\ 4) \end{cases}$ for $1 \le i \le n - \left\lceil \dfrac{n+2}{3} \right\rceil - 2$ when n is even and $f(v_i) = \begin{cases} 2 & \text{if } i \equiv 2, 3(mod\ 4) \\ 1 & \text{if } i \equiv 0, 1(mod\ 4) \end{cases}$ when n is odd; $f(v_i) = 1$ for $i = n - \left\lceil \dfrac{n+2}{3} \right\rceil - 1$ and $f(v_i) = 0$ if $n - \left\lceil \dfrac{n+2}{3} \right\rceil \le i \le n-1$, $f(v_n) = 1, f(v_1') = f(v_2') = 2$.

In view of the above labeling pattern we have $v_f(0) = v_f(1) = v_f(2) = \left\lceil \dfrac{n+2}{3} \right\rceil$, $e_f(0)-1 = e_f(1) = e_f(2) = \left\lceil \dfrac{n+2}{3} \right\rceil$.

For $n = 4$, we define the labeling as $f(v_1') = f(v_2') = 2$, $f(v_1) = f(v_2) = 0$ and $f(v_3) = f(v_4) = 1$.

Case 3 $n \equiv 2(mod\ 3), n > 5$.

Define $f(v_i) = \begin{cases} 2 & \text{if } i \equiv 2, 3(mod\ 4) \\ 1 & \text{if } i \equiv 0, 1(mod\ 4) \end{cases}$ for $1 \le i \le n - \left\lceil \dfrac{n+2}{3} \right\rceil$ when n is even and $f(v_i) = \begin{cases} 2 & \text{if } i \equiv 0, 1(mod\ 4) \\ 1 & \text{if } i \equiv 2, 3(mod\ 4) \end{cases}$ when n is odd; $f(v_i) = 0$ and $f(v_n) = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$ for $n - \left\lceil \dfrac{n+2}{3} \right\rceil < i \le n-1$, $f(v_1') = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$, $f(v_2') = \begin{cases} 2 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$

In view of the above labeling pattern we have $v_f(0) = v_f(1) - 1 = v_f(2) = \left\lfloor \frac{n+2}{3} \right\rfloor$ if n is even, $v_f(0) = v_f(1) = v_f(2) - 1 = \left\lfloor \frac{n+2}{3} \right\rfloor$ if n is odd and $e_f(0) = e_f(1) = e_f(2) + 1 = \left\lceil \frac{n+2}{3} \right\rceil$.

Thus in each case we have $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j = 0, 1, 2$. The duplicate $D(C_n, e')$ of a cycle C_n is a 3-product cordial graph. □

Example for 3-product cordial labeling of the graph G obtained by duplicating the edge v_1v_2 of the cycle C_8 is given in Fig.1.

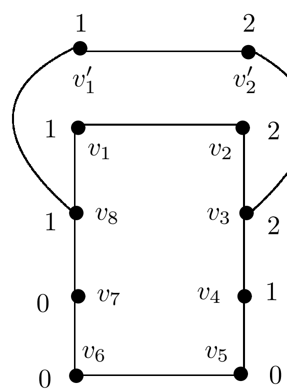

Fig.1

Theorem 2.2 *A duplicate graph $D(C_n, v')$ of a cycle C_n is a 3-product cordial graph.*

Proof Let $v_1, v_2, \ldots, v_n$ be the vertices of cycle C_n and let $v = v_1$ and the duplicated vertex is v_1'. Define a vertex labeling $f : V(G) \to \{0, 1, 2, \}$ by $f(v_i) = \begin{cases} 2 & \text{if } i \equiv 2, 3 (mod\ 4) \\ 1 & \text{if } i \equiv 0, 1 (mod\ 4) \end{cases}$ for $1 \leq i \leq n - \left\lceil \frac{n+2}{3} \right\rceil$, $f(v_i) = 0$ for $n - \left\lceil \frac{n+2}{3} \right\rceil < i \leq n - 1$ and $f(v_n) = 1, f(v_1') = 2$.

In view of the labeling pattern we have $v_f(0) = \frac{n}{3}$,

$$v_f(1) = \begin{cases} \frac{n}{3} + 1 & \text{if } n \text{ is odd} \\ \frac{n}{3} & \text{if } n \text{ is even} \end{cases}, \quad v_f(2) = \begin{cases} \frac{n}{3} + 1 & \text{if } n \text{ is even} \\ \frac{n}{3} & \text{if } n \text{ is odd} \end{cases}$$

Calculation shows that $e_f(0) = e_f(1) = e_f(2) + 1 = \left\lceil \frac{n+2}{3} \right\rceil$ if $n \equiv 0 (mod\ 3)$ and $v_f(0) + 1 = v_f(1) = v_f(2) = \left\lceil \frac{n+2}{3} \right\rceil$ and $e_f(0) = e_f(1) = e_f(2) = \left\lceil \frac{n+2}{3} \right\rceil$ if $n \equiv 1 (mod\ 3)$ and $v_f(0) = v_f(1) = v_f(2) = \left\lceil \frac{n+1}{3} \right\rceil$ and $e_f(0) - 1 = e_f(1) = e_f(2) = \left\lceil \frac{n+1}{3} \right\rceil$ if $n \equiv 2 (mod\ 3)$. Thus we have $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j = 0, 1, 2$. Hence the graph obtained by duplicating an arbitrary vertex of cycle $C_n (n > 4)$ admits 3-product cordial graph. If $n = 3$, we define the vertex labeling as $f(v_1') = 2, f(v_1) = 0$ and $f(v_2) = f(v_3) = 1$ and if $n = 4$, $f(v_1') = 2, f(v_1) = 1, f(v_2) = 2, f(v_3) = 0$ and $f(v_4) = 1$. Thus the result follows. □

Example for the 3-product labeling of the graph G obtained by duplicating the vertex v_1 of the cycle C_6 is given in Fig.2.

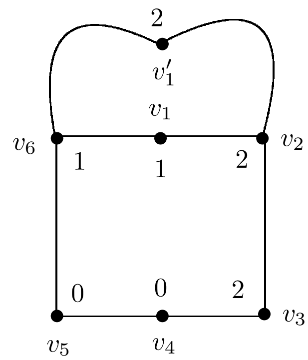

Fig.2

Theorem 2.3 *A graph obtained by duplication of arbitrary rim vertex of wheel W_n is 3-product cordial if $n \equiv 0, 2(mod\ 3)$.*

Proof Let $v_1, v_2, \ldots, v_n$ be the rim vertices of wheel W_n and let c be the apex vertex. Let G be the graph obtained by duplicating arbitrary rim vertex v_1 of wheel. Let v_1' be the duplicated vertex of v_1. Define a vertex labeling $f : V(G) \to \{0, 1, 2\}$ by considering the following two cases.

Case 1 $n \equiv 0(mod\ 3)$.

Define $f(v_i) = \begin{cases} 2 & \text{if } i \equiv 2, 3(mod\ 4) \\ 1 & \text{if } i \equiv 0, 1(mod\ 4) \end{cases}$ for $1 \leq i \leq 2\left\lceil \frac{n+2}{3} \right\rceil - 3$; $f(v_i) = 0$ for $2\left\lceil \frac{n+2}{3} \right\rceil - 2 \leq i \leq n-1$ and $f(v_n) = 1, f(v_1') = 2$; $f(c) = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even.} \end{cases}$

In view of the above labeling pattern we have $v_f(0)+1 = v_f(1) = v_f(2) = \left\lceil \frac{n+2}{3} \right\rceil$, $e_f(0) = e_f(1) = e_f(2) = \left\lceil \frac{2n+3}{3} \right\rceil$.

Case 2 $n \equiv 2(mod\ 3)$.

Define $f(v_i) = \begin{cases} 2 & \text{if } i \equiv 2, 3(mod\ 4) \\ 1 & \text{if } i \equiv 0, 1(mod\ 4) \end{cases}$ for $1 \leq i \leq 2\left\lfloor \frac{n+2}{3} \right\rfloor - 2$; $f(v_i) = 0$ for $2\left\lfloor \frac{n+2}{3} \right\rfloor - 1 \leq i \leq n-1$ and $f(v_n) = 1, f(v_1') = 2, f(c) = 1$.

In view of the above labeling pattern we have $v_f(0) = v_f(1)-1 = v_f(2) = \left\lfloor \frac{n+2}{3} \right\rfloor$, $e_f(0)-1 = e_f(1) = e_f(2) = \left\lfloor \frac{2n+3}{3} \right\rfloor$.

Thus in all the case we have $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j = 0, 1, 2$. Hence the graph obtained by duplicating of arbitrary rim vertex of wheel W_n is 3-product cordial if $n \equiv 0, 2(mod\ 3)$. □

Example for the 3-product cordial labeling of the graph G obtained by duplicating the vertex v_1 of the wheel W_9 is given in Fig.3.

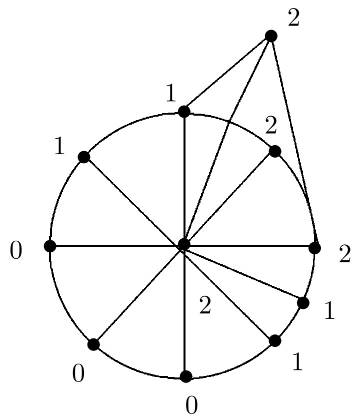

Fig.3

Theorem 2.4 *A ladder $L_n = P_n \times P_2$ is 3-product cordial.*

Proof Let the vertex set of $L_n = P_n \times P_2$ be $\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and the edge set of L_n be $\{u_i u_{i+1}/1 \leq i \leq n-1\} \cup \{v_i v_{i+1}/1 \leq i \leq n-1\} \cup \{u_i v_i/1 \leq i \leq n\}$. Then L_n has $2n$ vertices and $3n - 2$ edges.

Define a vertex labeling $f : V(L_n) \to \{0, 1, 2\}$ by considering the following three cases.

Case 1 $n \equiv 0(mod\ 3)$, let $n = 3m$.

$$\text{Define } f(u_i) = \begin{cases} 0 & \text{for } 1 \leq i \leq m \\ 2 & \text{for } i = m + j \text{ where } j \equiv 3(mod\ 4), 1 \leq j \leq 2m \\ 1 & \text{for otherwise} \end{cases}$$

$$f(v_i) = \begin{cases} 0 & \text{for } 1 \leq i \leq m \\ 1 & \text{for } i = m + j \text{ where } j \equiv 0(mod\ 4), 1 \leq j \leq 2m \\ 2 & \text{for otherwise} \end{cases}$$

In view of the above labeling pattern we have $v_f(0) = v_f(1) = v_f(2) = 2m$, $e_f(0) = 3m, e_f(1) = e_f(2) = 3m - 1$.

Case 2 $n \equiv 1(mod\ 3)$, let $n = 3m + 1$.

$$\text{Define } f(u_i) = \begin{cases} 0 & \text{for } 1 \leq i \leq m \\ 2 & \text{for } i = m + j \text{ where } j \equiv 3(mod\ 4), 1 \leq j \leq 2m \text{ and } f(u_n) = 1 \\ 1 & \text{for otherwise} \end{cases}$$

$$f(v_i)=\begin{cases}0 & \text{for } 1\le i\le m\\ 1 & \text{for } i=m+j \text{ where } j\equiv 0(mod\ 4), 1\le j\le 2m+1\\ 2 & \text{for otherwise}\end{cases}$$

In view of the above labeling pattern we have $v_f(0)=2m, v_f(1)=v_f(2)=2m+1, e_f(0)=3m, e_f(1)=\begin{cases}3m+1 & \text{if } m \text{ is odd}\\ 3m & \text{if } m \text{ is even}\end{cases}, e_f(2)=\begin{cases}3m+1 & \text{if } m \text{ is even}\\ 3m & \text{if } m \text{ is odd.}\end{cases}$

Case 3 $n\equiv 2(mod\ 3)$, let $n=3m+2$.

$$f(u_i)=\begin{cases}0 & \text{for } 1\le i\le m+1\\ 2 & \text{for } i=m+j \text{ where } j\equiv 3(mod\ 4), 2\le j\le 2m+2\\ 1 & \text{for otherwise}\end{cases}$$

$$f(v_i)=\begin{cases}0 & \text{for } 1\le i\le m\\ 1 & \text{for } i=m+j \text{ where } j\equiv 1(mod\ 4), 2\le j\le 2m+2,\\ 2 & \text{for otherwise}\end{cases}$$

$$f(v_{m+1})=\begin{cases}1 & \text{if } m \text{ is odd}\\ 2 & \text{if } m \text{ is even}\end{cases}$$

In view of the above labeling pattern we have $v_f(0)=v_f(1)=2m+1$, $v_f(2)=2m+2$ and $e_f(0)=3m+2, e_f(1)=e_f(2)=3m+1$. Thus in each case we have $|v_f(i)-v_f(j)|\le 1$ and $|e_f(i)-e_f(j)|\le 1$ for all $i,j=0,1,2$. Hence the graph L_n is 3-product cordial. $\square$

Example for 3-product cordial labeling of the graph G obtained by $L_7=P_7\times P_2$ is given in Fig.4.

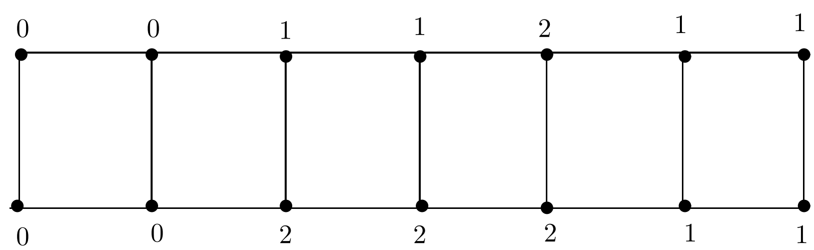

Fig.4

Theorem 2.5 *A triangular Ladder TL_n is 3-product cordial if $n\equiv 1,2(mod\ 3)$.*

Proof Let the vertex set of TL_n be $\{u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n\}$ and the edge set of TL_n be $\{u_iu_{i+1}/1\le i\le n-1\}\cup\{v_iv_{i+1}/1\le i\le n-1\}\cup\{u_iv_i/1\le i\le n\}\cup\{u_iv_{i+1}/1\le i\le n\}$. Then TL_n has $2n$ vertices and $4n-3$ edges.

Define a vertex labeling $f:V(L_n)\to\{0,1,2\}$ by considering the following two cases.

Case 1 $n\equiv 1(mod\ 3)$, let $n=3k+1$.

$$f(u_i)=\begin{cases}0 & \text{if } 1\le i\le k\\ 1 & \text{if } k+1\le i\le 3k+1\end{cases}, f(v_i)=\begin{cases}0 & \text{if } 1\le i\le k\\ 2 & \text{if } k+1\le i\le 3k+1\end{cases}$$

In view of the above labeling pattern we have $v_f(0)+1 = v_f(1) = v_f(2) = 2k+1, e_f(0) = e_f(1) = e_f(2) - 1 = 4k$.

Case 2 $n \equiv 2(mod\ 3)$, let $n = 3k+2$.

$$f(u_i) = \begin{cases} 0 & \text{if } 1 \leq i \leq k \\ 1 & \text{if } k+1 \leq i \leq 3k+1 \\ 0 & \text{if } i = 3k+2 \end{cases}, f(v_i) = \begin{cases} 0 & \text{if } 1 \leq i \leq k \\ 2 & \text{if } k+1 \leq i \leq 3k+2 \end{cases}$$

In view of the above labeling pattern we have $v_f(0) = v_f(1) = v_f(2) - 1 = 2k+1, e_f(0) = e_f(1) + 1 = e_f(2) = 4k+2$.

Thus in each case we have $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j = 0, 1, 2$. Hence the graph triangular ladder TL_n is 3-product cordial. □

Example for the 3-product cordial labeling of the graph G obtained by TL_8 is given in Fig.5.

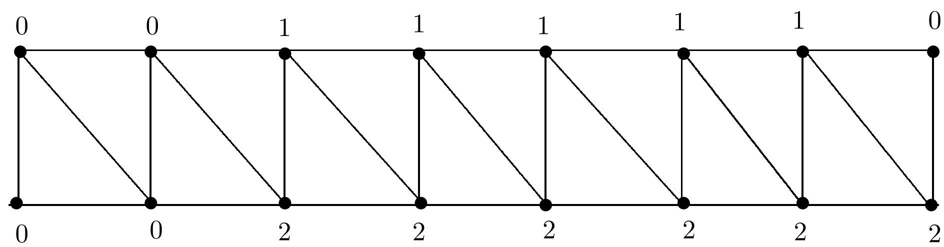

Fig.5

Theorem 2.6 *A graph* $\left\langle W_n^{(1)} : W_n^{(2)} : \cdots : W_n^{(k)} \right\rangle$ *is 3-product cordial if*

(*i*) *k is multiple of 3 and n is any positive integer;*

(*ii*) $k \equiv 1(mod\ 3)$ *and* $n \equiv 1(mod\ 3)$.

Proof Let $v_i^{(j)}$ be the rim vertices of $W_n^{(j)}$ and c_j be the apex vertices of $W_n^{(j)}$. Let $x_1, x_2, \ldots, x_{k-1}$ be the new vertices. The graph $G = \left\langle W_n^{(1)} : W_n^{(2)} : \cdots : W_n^{(k)} \right\rangle$ is obtained joining c_j with x_j and $x_{j+1}, 1 \leq j \leq k-1$. Define a vertex labeling $f : V(G) \to \{0, 1, 2\}$, we consider the following two cases.

Case 1 $k \equiv 0(mod\ 3, 1 \leq i \leq n$.

Define

$$f(v_i^{(j)}) = \begin{cases} 0 & \text{for } 1 \leq j \leq \frac{k}{3} \\ 2 & \text{for } \frac{k}{3} < j \leq \frac{2k}{3} \\ 1 & \text{for } \frac{2k}{3} < j \leq k \end{cases}, \qquad f(c_j) = \begin{cases} 0 & \text{for } 1 \leq j \leq \frac{k}{3} \\ 1 & \text{for } \frac{k}{3} < j \leq \frac{2k}{3} \\ 2 & \text{for } \frac{2k}{3} < j \leq k \end{cases}$$

and if $k = 3$, $f(x_j) = \begin{cases} 1 & \text{for } j = \left\lfloor \frac{k-1}{4} \right\rfloor + m \text{ where } m \equiv 1(mod\ 2), 1 \leq m \leq \frac{2k}{3} \\ 2 & \text{for otherwise.} \end{cases}$ and if

$$k > 3,\ f(x_j) = \begin{cases} 0 & \text{for } 1 \leq j \leq \left\lfloor \dfrac{k-1}{4} \right\rfloor \\ 1 & \text{for } j = \left\lfloor \dfrac{k-1}{4} \right\rfloor + m \text{ where } m \equiv 1(mod\ 2), 1 \leq m \leq \dfrac{2k}{3} \\ 2 & \text{for otherwise.} \end{cases}$$

In view of the above labeling pattern we have $v_f(0)+1 = v_f(1) = v_f(2) = \left\lceil \dfrac{nk+2k-1}{3} \right\rceil$, $e_f(0) = e_f(1) - 1 = e_f(2) = \left\lceil \dfrac{2nk+2k-2}{3} \right\rceil$.

Case 2 $n \equiv 1(mod\ 3), k \equiv 1(mod\ 3)$ and $1 \leq i \leq n$.

Let $n = 3k_1 + 1, k = 3k_2 + 1$ where k_1, k_2 are integer. Define $f(v_i^{(j)}) = 0, 1 \leq j \leq k_2$;

$$f(v_i^{(j)}) = \begin{cases} 0 & \text{for } 1 \leq i \leq k_1 \\ 1 & \text{for } i = k_1 + m \text{ where } m \equiv 2, 3(mod\ 4), 1 \leq m \leq 2k_1 + 1 \\ 2 & \text{for otherwise} \end{cases}$$

if $j = k_2 + 1$ when n is even and

$$f(v_i^{(j)}) = \begin{cases} 0 & \text{for } 1 \leq i \leq k_1 \\ 2 & \text{for } i = k_1 + m \text{ where } m \equiv 2, 3(mod\ 4), 1 \leq m \leq 2k_1 + 1 \\ 1 & \text{for otherwise} \end{cases}$$

when n is odd;

$$f(v_i^{(j)}) = \begin{cases} 2 & \text{for } k_2 + 1 < j \leq 2k_2 + 1 \\ 1 & \text{for } 2k_2 + 1 < j \leq 3k_2 + 1 \end{cases}$$

$$f(c_j) = \begin{cases} 0 & \text{for } 0 \leq j \leq k_2 \\ 2 & \text{for } j = k_2 + 1 \\ 1 & \text{for } k_2 + 1 < j \leq 2k_2 + 1 \\ 2 & \text{for otherwise} \end{cases}$$

$$f(x_j) = \begin{cases} 0 & \text{for } 1 \leq j \leq k_2 \\ 1 & \text{for } j = k_2 + m \text{ where } m \equiv 1(mod\ 2), 1 \leq m \leq 2k_2 \\ 2 & \text{for otherwise} \end{cases}$$

In view of the above labeling pattern we have $v_f(0)+1 = v_f(1) = v_f(2) = \left\lceil \dfrac{nk+2k-1}{3} \right\rceil$, $e_f(0) = e_f(1) + 1 = e_f(2) = \left\lceil \dfrac{2nk+2k-2}{3} \right\rceil$. Thus in each case we have $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j = 0, 1, 2$. Hence $\left\langle W_n^{(1)} : W_n^{(2)} : \cdots : W_n^{(k)} \right\rangle$ is 3-product cordial graph if (i) $k \equiv 0(mod\ 3), n$ is any value (ii) $k \equiv 1, n \equiv 1(mod\ 3)$. $\square$

Example for the 3-product cordial labeling of the graph $\left\langle W_n^{(1)} : W_n^{(2)} :\ W_n^{(3)} : \cdots : W_n^{(4)} \right\rangle$ is given in Fig.6.

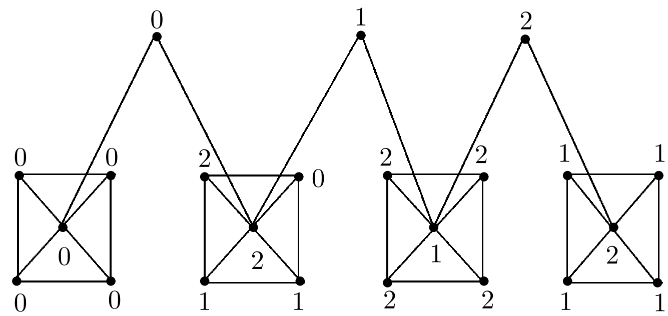

Fig.6

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The Line n-Sigraph of a Symmetric n-Sigraph-IV

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Abstract: An *n-tuple* $(a_1, a_2, \cdots, a_n)$ is *symmetric*, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, \cdots, a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. A*symmetric n-sigraph (symmetric n-marked graph)* is an ordered pair $S_n = (G, \sigma)$ ($S_n = (G, \mu)$), where $G = (V, E)$ is a graph called the *underlying graph* of S_n and $\sigma : E \to H_n$ ($\mu : V \to H_n$) is a function. In Bagga et al. (1995) introduced the concept of the *super line graph of index* r of a graph G, denoted by $\mathcal{L}_r(G)$. The vertices of $\mathcal{L}_r(G)$ are the r-subsets of $E(G)$ and two vertices P and Q are adjacent if there exist $p \in P$ and $q \in Q$ such that p and q are adjacent edges in G. Analogously, one can define the *super line symmetric n-sigraph of index* r of a symmetric n-sigraph $S_n = (G, \sigma)$ as a symmetric n-sigraph $\mathcal{L}_r(S_n) = (\mathcal{L}_r(G), \sigma')$, where $\mathcal{L}_r(G)$ is the underlying graph of $\mathcal{L}_r(S_n)$, where for any edge PQ in $\mathcal{L}_r(S_n)$, $\sigma'(PQ) = \sigma(P)\sigma(Q)$. It is shown that for any symmetric n-sigraph S_n, its $\mathcal{L}_r(S_n)$ is i-balanced and we offer a structural characterization of super line symmetric n-sigraphs of index r. Further, we characterize symmetric n-sigraphs S_n for which $S_n \sim \mathcal{L}_2(S_n)$, $\mathcal{L}_2(S_n) \sim L(S_n)$ and $\mathcal{L}_2(S_n) \sim \overline{S_n}$ where $\sim$ denotes switching equivalence and $\mathcal{L}_2(S_n)$, $L(S_n)$ and $\overline{S_n}$ are denotes the super line symmetric n-sigraph of index 2, line symmetric n-sigraph and complementary symmetric n-sigraph of S_n respectively. Also, we characterize symmetric n-sigraphs S_n for which $S_n \cong \mathcal{L}_2(S_n)$ and $\mathcal{L}_2(S_n) \cong L(S_n)$.

Key Words: Smarandachely symmetric n-marked graph, symmetric n-sigraph, symmetric n-marked graph, balance, switching, balance, super line symmetric n-sigraph, line symmetric n-sigraph.

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§1. Introduction

Unless mentioned or defined otherwise, for all terminology and notion in graph theory the reader is refer to [6]. We consider only finite, simple graphs free from self-loops.

Let $n \geq 1$ be an integer. An *n-tuple* $(a_1, a_2, \cdots, a_n)$ is *symmetric*, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, \cdots, a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. Note that H_n is a group under coordinate wise multiplication, and the

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order of H_n is 2^m, where $m = \lceil \frac{n}{2} \rceil$.

A *Smarandachely k-marked graph* is an ordered pair $S = (G, \mu)$ where $G = (V, E)$ is a graph called *underlying graph of S* and $\mu : V \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$ is a function, where each $\overline{e}_i \in \{+, -\}$. An *n-tuple* $(a_1, a_2, ..., a_n)$ is *symmetric*, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, ..., a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. A *Smarandachely symmetric n-marked graph* is an ordered pair $S_n = (G, \mu)$, where $G = (V, E)$ is a graph called the *underlying graph* of S_n and $\mu : V \to H_n$ is a function. Particularly, a Smarandachely 2-marked graph is called a *symmetric n-sigraph (symmetric n-marked graph)*, where $G = (V, E)$ is a graph called the *underlying graph* of S_n and $\sigma : E \to H_n$ ($\mu : V \to H_n$) is a function.

In this paper by an *n-tuple/n-sigraph/n-marked graph* we always mean a symmetric n-tuple/symmetric n-sigraph/symmetric n-marked graph.

An n-tuple $(a_1, a_2, \cdots, a_n)$ is the *identity n-tuple*, if $a_k = +$, for $1 \leq k \leq n$, otherwise it is a *non-identity n-tuple*. In an n-sigraph $S_n = (G, \sigma)$ an edge labelled with the identity n-tuple is called an *identity edge*, otherwise it is a *non-identity edge*.

Further, in an n-sigraph $S_n = (G, \sigma)$, for any $A \subseteq E(G)$ the *n-tuple* $\sigma(A)$ is the product of the n-tuples on the edges of A.

In [12], the authors defined two notions of balance in n-sigraph $S_n = (G, \sigma)$ as follows (See also R. Rangarajan and P.S.K.Reddy [9]:

Definition 1.1 *Let $S_n = (G, \sigma)$ be an n-sigraph. Then,*

(*i*) *S_n is* identity balanced *(or* i-balanced*), if product of n-tuples on each cycle of S_n is the identity n-tuple, and*

(*ii*) *S_n is* balanced*, if every cycle in S_n contains an even number of non-identity edges.*

Note An i-balanced n-sigraph need not be balanced and conversely.

The following characterization of i-balanced n-sigraphs is obtained in [12].

Proposition 1.1(E. Sampathkumar et al. [12]) *An n-sigraph $S_n = (G, \sigma)$ is i-balanced if, and only if, it is possible to assign n-tuples to its vertices such that the n-tuple of each edge uv is equal to the product of the n-tuples of u and v.*

In [12], the authors also have defined switching and cycle isomorphism of an n-sigraph $S_n = (G, \sigma)$ as follows (See also [7,10,11] & [14]-[18]):

Let $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$, be two n-sigraphs. Then S_n and S'_n are said to be *isomorphic*, if there exists an isomorphism $\phi : G \to G'$ such that if uv is an edge in S_n with label $(a_1, a_2, \cdots, a_n)$ then $\phi(u)\phi(v)$ is an edge in S'_n with label $(a_1, a_2, \cdots, a_n)$.

Given an n-marking μ of an n-sigraph $S_n = (G, \sigma)$, *switching* S_n with respect to μ is the operation of changing the n-tuple of every edge uv of S_n by $\mu(u)\sigma(uv)\mu(v)$. The n-sigraph obtained in this way is denoted by $\mathcal{S}_\mu(S_n)$ and is called the *μ-switched n-sigraph* or just *switched n-sigraph*.

Further, an n-sigraph S_n *switches* to n-sigraph S'_n (or that they are *switching equivalent*

to each other), written as $S_n \sim S'_n$, whenever there exists an n-marking of S_n such that $\mathcal{S}_\mu(S_n) \cong S'_n$.

Two n-sigraphs $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$ are said to be *cycle isomorphic*, if there exists an isomorphism $\phi : G \to G'$ such that the n-tuple $\sigma(C)$ of every cycle C in S_n equals to the n-tuple $\sigma(\phi(C))$ in S'_n. We make use of the following known result (see [12]).

Proposition 1.2(E. Sampathkumar et al. [12]) *Given a graph G, any two n-sigraphs with G as underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

In this paper, we introduced the notion called super line n-sigraph of index r and we obtained some interesting results in the following sections. The super line n-sigraph of index r is the generalization of line n-sigraph.

§2. Super Line n-Sigraph $\mathcal{L}_r(S_n)$

In [1], the authors introduced the concept of the *super line graph*, which generalizes the notion of line graph. For a given G, its super line graph $\mathcal{L}_r(G)$ of index r is the graph whose vertices are the r-subsets of $E(G)$, and two vertices P and Q are adjacent if there exist $p \in P$ and $q \in Q$ such that p and q are adjacent edges in G. In [1], several properties of $\mathcal{L}_r(G)$ were studied. Many other properties and concepts related to super line graphs were presented in [2,4]. The study of super line graphs continues the tradition of investigating generalizations of line graphs in particular and of graph operators in general, as elaborated in the classical monograph by Prisner [8]. From the definition, it turns out that $\mathcal{L}_1(G)$ coincides with the line graph $L(G)$. More specifically, some results regarding the super line graph of index 2 were presented in [3] and [5]. Several variations of the super line graph have been considered.

In this paper, we extend the notion of $\mathcal{L}_r(G)$ to realm of n-sigraphs as follows: The *super line n-sigraph of index* r of an n-sigraph $S_n = (G, \sigma)$ as an n-sigraph $\mathcal{L}_r(S_n) = (\mathcal{L}_r(G), \sigma')$, where $\mathcal{L}_r(G)$ is the underlying graph of $\mathcal{L}_r(S_n)$, where for any edge PQ in $\mathcal{L}_r(S_n)$, $\sigma'(PQ) = \sigma(P)\sigma(Q)$.

Hence, we shall call a given n-sigraph S_n a *super line n-sigraph of index* r if it is isomorphic to the super line n-sigraph of index r, $\mathcal{L}_r(S'_n)$ of some n-sigraph S'_n. In the following subsection, we shall present a characterization of super line n-sigraph of index r.

The following result indicates the limitations of the notion $\mathcal{L}_r(S_n)$ as introduced above, since the entire class of i-unbalanced n-sigraphs is forbidden to be super line n-sigraphs of index r.

Proposition 2.1 *For any n-sigraph $S_n = (G, \sigma)$, its $\mathcal{L}_r(S_n)$ is i-balanced.*

Proof Let σ' denote the n-tuple of $\mathcal{L}_r(S_n)$ and let the n-tuple σ of S_n be treated as an n-marking of the vertices of $\mathcal{L}_r(S_n)$. Then by definition of $\mathcal{L}_r(S_n)$ we see that $\sigma'(PQ) = \sigma(P)\sigma(Q)$, for every edge PQ of $\mathcal{L}_r(S_n)$ and hence, by Proposition 1.1, the result follows. □

Corollary 2.2 *For any n-sigraph $S_n = (G, \sigma)$, its $\mathcal{L}_2(S_n)$ is i-balanced.*

For any positive integer k, the k^{th} iterated super line n-sigraph of index r, $\mathcal{L}_r(S_n)$ of S_n is defined as follows:

$$\mathcal{L}_r^0(S_n) = S_n,\ \mathcal{L}_r^k(S_n) = \mathcal{L}_r(\mathcal{L}_r^{k-1}(S_n))$$

Corollary 2.3 *For any n-sigraph $S_n = (G, \sigma)$ and any positive integer k, $\mathcal{L}_r^k(S_n)$ is i-balanced.*

The *line graph* $L(G)$ of graph G has the edges of G as the vertices and two vertices of $L(G)$ are adjacent if the corresponding edges of G are adjacent. The *line n-sigraph* of an n-sigraph $S_n = (G, \sigma)$ is an n-sigraph $L(S_n) = (L(G), \sigma')$, where for any edge ee' in $L(S_n)$, $\sigma'(ee') = \sigma(e)\sigma(e')$. This concept was introduced by E. Sampatkumar et al. [13]. The following result is one can easily deduce from Proposition 2.1.

Corollary 2.4 (E. Sampathkumar et al. [13]) *For any n-sigraph $S_n = (G, \sigma)$, its line n-sigraph $L(S_n)$ is i-balanced.*

In [5], the authors characterized those graphs that are isomorphic to their corresponding super line graphs of index 2.

Proposition 2.5(K. S. Bagga et al. [5]) *For a graph $G = (V, E)$, $G \cong \mathcal{L}_2(G)$ if, and only if, $G = K_3$.*

We now characterize the n-sigraphs that are switching equivalent to their super line n-sigraphs of index 2.

Proposition 2.6 *For any n-sigraph $S_n = (G, \sigma)$, $S_n \sim \mathcal{L}_2(S_n)$ if, and only if, $G = K_3$ and S is i-balanced n-sigraph.*

Proof Suppose $S_n \sim \mathcal{L}_2(S_n)$. This implies, $G \cong \mathcal{L}_2(G)$ and hence G is K_3. Now, if S_n is any n-sigraph with underlying graph as K_3, Corollary 2.2 implies that $\mathcal{L}_2(S_n)$ is i-balanced and hence if S_n is i-unbalanced and its $\mathcal{L}_2(S_n)$ being i-balanced can not be switching equivalent to S_n in accordance with Proposition 1.2. Therefore, S_n must be i-balanced.

Conversely, suppose that S_n is i-balanced n-sigraph and G is K_3. Then, since $\mathcal{L}_2(S_n)$ is i-balanced as per Corollary 2.2 and since $G \cong \mathcal{L}_2(G)$, the result follows from Proposition 1.2 again. □

We now characterize the n-sigraphs that are isomorphic to their super line n-sigraphs of index 2.

Proposition 2.7 *For any n-sigraph $S_n = (G, \sigma)$, $S_n \cong \mathcal{L}_2(S_n)$ if, and only if, $G = K_3$ and S_n is i-balanced n-sigraph.*

In [5], the authors characterized whose super line graphs of index 2 that are isomorphic to $L(G)$.

Proposition 2.8(K. S. Bagga et al. [5]) *For a graph $G = (V, E)$, $\mathcal{L}_2(G) \cong L(G)$ if, and only if, G is $K_{1,3}$, K_3 or $3K_2$.*

From the above result we have following result for signed graphs:

Proposition 2.9 *For any n-sigraph $S_n = (G, \sigma)$, $\mathcal{L}_2(S_n) \sim L(S_n)$ if, and only if, G is $K_{1,3}$, K_3 or $3K_2$.*

Proof Suppose $\mathcal{L}_2(S_n) \sim L(S_n)$. This implies, $\mathcal{L}_2(G) \cong L(G)$ and hence by Proposition 2.8, we see that the graph G must be isomorphic to $K_{1,3}$, K_3 or $3K_2$.

Conversely, suppose that G is a $K_{1,3}$, K_3 or $3K_2$. Then $\mathcal{L}_2(G) \cong L(G)$ by Proposition 2.8. Now, if S_n any n-sigraph on any of these graphs, By Proposition 2.1 and Corollary 2.4, $\mathcal{L}_2(S_n)$ and $L(S_n)$ are i-balanced and hence, the result follows from Proposition 1.2. □

We now characterize n-sigraphs whose super line n-sigraphs $\mathcal{L}_2(S_n)$ that are isomorphic to line n-sigraphs.

Proposition 2.10 *For any n-sigraph $S_n = (G, \sigma)$, $\mathcal{L}_2(S_n) \cong L(S_n)$ if, and only if, G is $K_{1,3}$, K_3 or $3K_2$.*

Proof Clearly $\mathcal{L}_2(G) \cong L(G)$, when G is $K_{1,3}$, K_3 or $3K_2$. Consider the map $f : V(\mathcal{L}_2(G)) \rightarrow V(L(G))$ defined by $f(e_1e_2, e_2e_3) = (e_1, e_3)$ is an isomorphism. Let σ be any n-tuple on $K_{1,3}$, K_3 or $3K_2$. Let $e = (e_1e_2, e_2e_3)$ be an edge in $\mathcal{L}_2(G)$, where G is $K_{1,3}$, K_3 or $3K_2$. Then the n-tuple of the edge e in $\mathcal{L}_2(G)$ is the $\sigma(e_1e_2)\sigma(e_2e_3)$ which is the n-tuple of the edge (e_1, e_3) in $L(G)$, where G is $K_{1,3}$, K_3 or $3K_2$. Hence the map f is also an n-sigraph isomorphism between $\mathcal{L}_2(S_n)$ and $L(S_n)$. □

Let $S_n = (G, \sigma)$ be an n-sigraph. *The complement* of S_n is an n-sigraph $\overline{S_n} = (\overline{G}, \sigma^c)$, where $\overline{G}$ is the underlying graph of $\overline{S_n}$ and for any edge $e = uv \in \overline{S_n}$, $\sigma^c(uv) = \mu(u)\mu(v)$, where for any $v \in V$, $\mu(v) = \prod_{u \in N(v)} \sigma(uv)$. Clearly, $\overline{S_n}$ as defined here is an i-balanced n-sigraph due to Proposition 1.1.

In [5], the authors proved there are no solutions to the equation $\mathcal{L}_2(G) \sim \overline{G}$. So it is impossible to construct switching equivalence relation of $\mathcal{L}_2(S_n) \sim \overline{S_n}$ for any arbitrary n-sigraph. The following result characterizes n-sigraphs which are super line n-sigraphs of index r.

Proposition 2.11 *An n-sigraph $S_n = (G, \sigma)$ is a super line n-sigraph of index r if and only if S_n is i-balanced n-sigraph and its underlying graph G is a super line graph of index r.*

Proof Suppose that S_n is i-balanced and G is a $\mathcal{L}_r(G)$. Then there exists a graph H such that $\mathcal{L}_r(H) \cong G$. Since S_n is i-balanced, by Proposition 1.1, there exists an n-marking μ of G such that each edge uv in S_n satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the n-sigraph $S'_n = (H, \sigma')$, where for any edge e in H, $\sigma'(e)$ is the n-marking of the corresponding vertex in G. Then clearly, $\mathcal{L}_r(S'_n) \cong S_n$. Hence S_n is a super line n-sigraph of index r.

Conversely, suppose that $S_n = (G, \sigma)$ is a super line n-sigraph of index r. Then there exists an n-sigraph $S'_n = (H, \sigma')$ such that $\mathcal{L}_r(S'_n) \cong S_n$. Hence G is the $\mathcal{L}_r(G)$ of H and by Proposition 2.1, S_n is i-balanced. □

If we take $r = 1$ in $\mathcal{L}_r(S_n)$, then this is the ordinary line n-sigraph. In [13], the authors obtained structural characterization of line n-sigraphs and clearly Proposition 2.11 is the generalization of line n-sigraphs.

Proposition 2.12(E. Sampathkumar et al. [13]) *An n-sigraph $S_n = (G, \sigma)$ is a line n-sigraph if, and only if, S_n is i-balanced n-sigraph and its underlying graph G is a line graph.*

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Imagination is not to be divorced from the facts: it is a way of illuminating the facts.

By Alfred North Whitehead, a British philosopher and mathematician.

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